Linear potential theory for tsunami generation and propagation

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Running Title: Tsunami generation and propagation

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Keywords: tsunami, theory, generation and propagation

Original version was submitted on November 5, 2012
Linear potential theory

A general framework

We formulate the tsunami generation and propagation from the sea-bottom deformation in a constant water depth based on a linear potential theory [e.g., Takahashi 1942; Hammack 1973]. We use the Cartesian coordinates shown in Figure 1, where the $z$-axis is vertically upward, and the $x$- and $y$-axes in a horizontal plane. The sea surface is located at $z = 0$, and the sea bottom is flat and located at $z = -h_0$. We assume that the height of the water surface $\eta(x,y,t)$ at time $t$ is small enough compared to the water depth $|\eta| < h_0$ and the viscosity is neglected. The velocity in the fluid is given by a vector $v(x,t) = v_x e_x + v_y e_y + v_z e_z$, where $x = x e_x + y e_y + z e_z$, and $e_x$, $e_y$, and $e_z$ are the basis vectors in the $x$, $y$, and $z$ axes, respectively. We also assume an incompressible and irrotational flow, $\nabla \cdot v = 0$, in which the velocity vector is given as $v(x,t) = \nabla \phi(x,t)$ using a velocity potential $\phi(x,t)$.

The velocity potential satisfies the Laplace equation

$$\Delta \phi(x,t) = 0,$$  \hspace{1cm} (1)

and the boundary conditions at the surface ($z = 0$) are given by

$$\frac{\partial \phi(x,t)}{\partial t} \bigg|_{z=0} + g_0 \eta(x,y,t) = 0,$$  \hspace{1cm} (2)

$$\frac{\partial \phi(x,t)}{\partial z} \bigg|_{z=0} - \frac{\partial \eta(x,y,t)}{\partial t} = 0,$$  \hspace{1cm} (3)
where $g_0$ is the gravitational constant. Assuming the final sea-bottom deformation, or permanent vertical displacement at the sea bottom, to be $d(x,y)$, we give the vertical component of the velocity as the boundary condition at the sea bottom, as follows:

$$v_z(x,t)|_{z=-h_0} = d(x,y) \chi(t),$$  \hspace{1cm} (4)

where the function $\chi(t)$ depends only on time, which satisfies the following:

$$\int_{-\infty}^{\infty} \chi(t) dt = 1.$$  \hspace{1cm} (5)

The function of $\chi(t)$ has the dimension of the inverse of time and is referred to hereinafter as the source time function. We obtained the velocity potential that satisfies (1), (2), (3), and (4) as follows [e.g., Saito and Furmura 2009]:

$$\phi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \exp[-i\omega t] \hat{\chi}(\omega)$$

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_x dk_y \exp[ik_x x + ik_y y] \frac{1}{k} \frac{\omega^2 \sinh k z + g_0 k \cosh k z}{\omega^2 \cosh h_0 - g_0 k \sinh h_0} \tilde{d}(k_x, k_y),$$  \hspace{1cm} (6)

where $k = \sqrt{k_x^2 + k_y^2}$, $\hat{\chi}(\omega)$ is the Fourier transform in the time-frequency domain.
defined as follows:

\[ \hat{\chi}(\omega) = \int_{-\infty}^{\infty} d\tau \exp[i\omega \tau] \chi(\tau) \]  

(7)

and \( \tilde{d}(k_x, k_y) \) is the 2-D Fourier transform in space-wavenumber domain defined as follows:

\[ \tilde{d}(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy d(x, y) \exp[-i(k_x x + k_y y)]. \]  

(8)

Eq. (6) represents a formal expression of the velocity potential with the inverse Fourier transform with respect to the time-frequency domain. This was derived by Takahashi [1942] in cylindrical coordinates. Similar equations were obtained by Kervella et al. [2007] and Levin and Nosov [2009] in the Cartesian coordinates using the inverse Laplace transform. It is necessary to conduct an integration over the angular frequency \( \omega \) in order to obtain a solution in the time domain. Takahashi [1942] and Kervell et al. [2007] calculated the integral for the sea surface \( z = 0 \), but not for an arbitrary depth, \( z \). Levin and Nosov [2009] performed the inverse Laplace transform for the velocity potential for any depth \( z \). However, they assumed a very special case with the boundary condition at the sea bottom given by a linearly increasing sea-bottom deformation for 1-D sea-bottom deformation (Eqs. (2.67) and (2.68) in Levin and Nosov [2009]). A solution for the integration over the angular frequency \( \omega \) in Eq. (6) has not yet been obtained for the sea-bottom deformation generally given by Eq. (4).
The main difficulty with respect to the integration is that the residue theory is not applicable to the sea-bottom deformation that is given by the arbitrary function of \( \chi(t) \) or \( \chi(t) = \delta(t) \). In the following, we theoretically derive the solution in the time domain for the sea-bottom deformation that is given by the arbitrary function of \( \chi(t) \).

A general solution: impulse response

We obtained the solution for the instantaneous sea-bottom deformation or for the impulse response of the source time function given by \( \chi(t) = \delta(t) \). The solution is obtained as follows:

\[
\phi_{\text{impulse}}(x,t) = \left\{ \begin{array}{ll}
-\frac{1}{(2\pi)^2} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \exp[ik_x x + ik_y y] \tilde{d}(k_x, k_y) \\
\times & \frac{1}{k} \left[ \frac{\cosh kz}{\sinh kh_0} \delta(t) + \left( \frac{\cosh kz}{\sinh kh_0} + \frac{\sinh kz}{\cosh kh_0} \right) \omega_0 \sin \omega_0 t \cdot H(t) \right] \\
- & \left( \frac{\cosh kz}{\sinh kh_0} + \frac{\sinh kz}{\cosh kh_0} \right) \cos \omega_0 t \cdot \delta(t) \right\}
\end{array} \right. ,
\]

(17)

where

\[
H(t) = \left\{ \begin{array}{ll}
1 & \text{for } 0 \leq t \leq T \\
0 & \text{for } t < 0, t > T
\end{array} \right. .
\]

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By convoluting an arbitrary function of $\chi(t)$ with Eq. (17), we now obtain the solution of the velocity potential with respect to the function generally given by Eq. (4), as follows:

$$\phi(x,t) = \int_{-\infty}^{\infty} \phi_{\text{impulse}}(x,t-\tau) \chi(\tau) d\tau$$  \hspace{1cm} (18)

Using Eq. (17), we obtain the horizontal components of the velocity field $v_H = v_x e_x + v_y e_y$, the vertical component of the velocity $v_z$, and the height of the water surface $\eta$ for an instantaneous sea-bottom deformation, or the velocity at the bottom is given by the delta function, $v_z(x,t)_{z=-h_0} = d(x,y) \delta(t)$ as follows:

$$v_H(x,t) = \nabla_H \phi_{\text{impulse}}(x,t)$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \exp[i k_x x + i k_y y] \frac{\tilde{d}(k_x,k_y)}{\cosh kh_0} \times \left\{ \frac{-i g_0 k_H}{\omega_0} f_{\eta}(k,z,h_0) \sin \omega_0 t \cdot H(t) + \frac{i k_H}{k} \sinh k z \cdot \delta(t) \right\}$$

$$v_z(x,t) = \frac{\partial \phi_{\text{impulse}}(x,t)}{\partial z}$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \exp[i k_x x + i k_y y] \frac{\tilde{d}(k_x,k_y)}{\cosh kh_0} \times \left\{ -\omega_0 f_z(k,z,h_0) \sin \omega_0 t \cdot H(t) + \cosh k z \cdot \delta(t) \right\}$$  \hspace{1cm} (20)

$$\omega_0 = \sqrt{g_0 k \tanh kh_0}.$$
\[ \eta(x,y,t) = -\frac{1}{g_0} \left. \frac{\partial \phi_{\text{impulse}}(x,t)}{\partial t} \right|_{t=0} \]
\[ = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \exp\left[i k_x x + ik_y y\right] \frac{\bar{d}(k_x, k_y)}{\cosh k h_0} \cos \omega t \cdot H(t) \]  
(22)

where \( \nabla_H \) is the gradient in the horizontal plane given by \( \nabla_H = \partial/\partial x \mathbf{e}_x + \partial/\partial y \mathbf{e}_y \), and \( \mathbf{k}_H \) is the wavenumber vector in the horizontal plane given by \( \mathbf{k}_H = k_x \mathbf{e}_x + k_y \mathbf{e}_y \). Here, we introduced the distribution functions of the horizontal and vertical components of the velocity, as follows:

\[ f_{\text{H}}(k_z, h_0) = \cosh k z + \tanh k h_0 \sinh k z , \]  
(23)

\[ f_{\text{z}}(k_z, h_0) = \frac{\sinh k z}{\tanh k h_0} + \cosh k z . \]  
(24)

We discuss the meaning of these distribution functions (Eqs. (23) and (24)) together with the interpretations of Eqs. (20) through (22) in the following section. We can confirm that Eqs. (20) through (22) satisfy \( \nabla \cdot \mathbf{v} = 0 \) and the boundary conditions (Eqs. (2) through (4)).

It is also beneficial to provide a representation for the pressure at the sea bottom because an ocean-bottom pressure gauge has been often used for recording tsunami generation and propagation [e.g., Tang et al. 2011; Tsushima et al. 2012]. The pressure in the ocean is given by the sum of the hydrostatic pressure and an excess pressure due to the wave motion, i.e., \( -\rho_0 g_0 z + p_e \), where the excess pressure is given by the velocity
potential as \( p_e = -\rho_0 \frac{\partial \phi}{\partial t} \). Using the velocity potential of Eq. (19), we obtain the excess pressure brought about by the tsunami at the sea bottom as follows:

\[
\bar{p}_e(x, t) \bigg|_{z=-h_0} = -\rho_0 \frac{\partial \bar{\phi}(x, t)}{\partial t} \bigg|_{z=-h_0} = \frac{1}{(2\pi)^2} \int \int d^2 k \exp \left[ i \left( k_x x + k_y y \right) \right] \times \frac{\rho_0}{\cosh kh_0} \left\{ g_0 \left( \frac{\cos \omega_0 t}{\cosh kh_0} \right) H(t) + \frac{1}{k} \sinh kh_0 \frac{d \delta(t)}{dt} \right\},
\]

for a point impulse response. For the source given by (4), we obtain,

\[
p_e(x, t) \bigg|_{z=-h_0} = \frac{1}{(2\pi)^2} \int \int d^2 k \exp \left[ i \left( k_x x + k_y y \right) \right] \times \frac{\rho_0 \bar{d}(k_x, k_y)}{\cosh kh_0} \left\{ g_0 \int \cos \left[ \omega_0 (t - \tau) \right] \chi(\tau) d\tau + \frac{1}{k} \sinh kh_0 \frac{d \chi(t)}{dt} \right\}.
\]

This equation is similar but not identical to that obtained by Kervell et al. [2007].

Kervella et al. [2007] derived the pressure at the sea bottom after an instantaneous sea-bottom uplift. However, they did not consider a source term, so that the pressure change during the source process time cannot be inclusive. On the other hand, Eq. (26) includes an additional term (the second term in brackets \( \{ \cdot \} \) in Eq. (26)) as a source term, which represents the contribution from the source. This term leads to an increase in excess pressure at the sea bottom when the sea-bottom uplifts with an increasing rate \( d\chi/dt > 0 \).
References

Cui, H., J.D. Pietrzak, G.S. Stelling (2010), A finite volume analogue of the $P_1^{NC}$-$P_1$ finite element: With accurate flooding and drying, Ocean Modeling 35, 16-30, doi:10.1016/j.ocemod.2010.06.001.


Kajiura, K. (1963), The leading wave of a tsunami, *Bulletin of the Earthquake Research Institute* 41, 545-571.


Figure 1. Coordinates used for the formulation.