Derivation of Tables 7 through 9 in Okada (1992)

[I] Derivation of Table 7 (x-Derivative)

Table 7 can be derived by differentiation of Table 6 with x-coordinate.
In the following, the notation is matched with Tables in Okada (1992).

Displacement :
$$u_x(x,y,z) = \frac{U}{2\pi}\left[\ u_1{}^A - \hat{u}_1{}^A + u_1{}^B + z u_1{}^C\ \right]$$
$$u_y(x,y,z) = \frac{U}{2\pi}\left[(u_2{}^A - \hat{u}_2{}^A + u_2{}^B + z u_2{}^C)\cos\delta - (u_3{}^A - \hat{u}_3{}^A + u_3{}^B + z u_3{}^C)\sin\delta\,\right]$$
$$u_z(x,y,z) = \frac{U}{2\pi}\left[(u_2{}^A - \hat{u}_2{}^A + u_2{}^B - z u_2{}^C)\sin\delta + (u_3{}^A - \hat{u}_3{}^A + u_3{}^B - z u_3{}^C)\cos\delta\,\right]$$
$$u_i{}^A = \ f_i{}^A(\xi,\eta,z)|_{\xi=x}^{\xi=x-L}.|_{\eta=p}^{\eta=p-W},\quad \hat{u}_i{}^A = f_i{}^A(\xi,\eta,-z)\|,\quad u_i{}^B = f_i{}^B(\xi,\eta,z)\|,\quad u_i{}^C = f_i{}^C(\xi,\eta,z)\|$$

x-Derivative :
$$\frac{\partial u_x}{\partial x}(x,y,z) = \frac{U}{2\pi}\left[\,j_1{}^A - \hat{j}_1{}^A + j_1{}^B + z j_1{}^C\,\right]$$
$$\frac{\partial u_y}{\partial x}(x,y,z) = \frac{U}{2\pi}\left[(j_2{}^A - \hat{j}_2{}^A + j_2{}^B + z j_2{}^C)\cos\delta - (j_3{}^A - \hat{j}_3{}^A + j_3{}^B + z j_3{}^C)\sin\delta\,\right]$$
$$\frac{\partial u_z}{\partial x}(x,y,z) = \frac{U}{2\pi}\left[(j_2{}^A - \hat{j}_2{}^A + j_2{}^B - z j_2{}^C)\sin\delta + (j_3{}^A - \hat{j}_3{}^A + j_3{}^B - z j_3{}^C)\cos\delta\,\right]$$
$$j_i{}^A = \ \partial f_i{}^A/\partial x(\xi,\eta,z)|_{\xi=x}^{\xi=x-L}.|_{\eta=p}^{\eta=p-W},\quad \hat{j}_i{}^A = \ \partial f_i{}^A/\partial x(\xi,\eta,-z)\|,\quad j_i{}^B = \ \partial f_i{}^B/\partial x(\xi,\eta,z)\|,\quad j_i{}^C = \ \partial f_i{}^C/\partial x(\xi,\eta,z)\|$$

(1) Strike slip

$$f_i{}^A = \begin{pmatrix} u_1 = \dfrac{\Theta}{2} & +\dfrac{\alpha}{2}\xi q Y_{11} \\ u_2 = & \dfrac{\alpha}{2}\dfrac{q}{R} \\ u_3 = \dfrac{1-\alpha}{2}\ln(R+\eta) & -\dfrac{\alpha}{2}q^2 Y_{11} \end{pmatrix} \qquad f_i{}^B = \begin{pmatrix} u_1 = -\xi q Y_{11} - \Theta & -\dfrac{1-\alpha}{\alpha}I_1\sin\delta \\ u_2 = -\dfrac{q}{R} & +\dfrac{1-\alpha}{\alpha}\dfrac{\tilde{y}}{R+\tilde{d}}\sin\delta \\ u_3 = q^2 Y_{11} & -\dfrac{1-\alpha}{\alpha}I_2\sin\delta \end{pmatrix}$$
$$\Theta = \tan^{-1}\frac{\xi\eta}{qR} \qquad Y_{11} = \frac{1}{R(R+\eta)}$$

$$f_i{}^C = \begin{pmatrix} u_1 = (1-\alpha)\xi Y_{11}\cos\delta & -\alpha\xi q Z_{32} \\ u_2 = (1-\alpha)\left(\dfrac{\cos\delta}{R} + 2qY_{11}\sin\delta\right) & -\alpha\dfrac{\tilde{c}q}{R^3} \\ u_3 = (1-\alpha)qY_{11}\cos\delta & -\alpha\left(\dfrac{\tilde{c}\eta}{R^3} - zY_{11} + \xi^2 Z_{32}\right) \end{pmatrix}$$
$$Y_{32} = \frac{2R+\eta}{R^3(R+\eta)^2} \qquad Z_{32} = \frac{\sin\delta}{R^3} - hY_{32}$$
$$h = q\cos\delta - z\ ,\qquad \tilde{c} = \tilde{d} + z = \eta\sin\delta - h$$

where, $q = y\sin\delta - d\cos\delta,\quad \tilde{y} = \eta\cos\delta + q\sin\delta,\quad \tilde{d} = \eta\sin\delta - q\cos\delta,\quad R^2 = \xi^2 + \eta^2 + q^2 = X^2 + \eta^2\,,$

$$I_1 = -\frac{\xi}{R+\tilde{d}}\cos\delta - I_4\sin\delta,\qquad I_2 = \ln\left(R+\tilde{d}\right) + I_3\sin\delta$$
$$I_3 = \frac{1}{\cos\delta}\frac{\tilde{y}}{R+\tilde{d}} - \frac{1}{\cos^2\delta}\left[\ln(R+\eta) - \sin\delta\ln(R+\tilde{d})\right] \qquad \left(I_3 = \frac{1}{2}\left[\frac{\eta}{R+\tilde{d}} + \frac{\tilde{y}q}{(R+\tilde{d})^2} - \ln(R+\eta)\right]\quad \text{if } \cos\delta = 0\right)$$
$$I_4 = \frac{\sin\delta}{\cos\delta}\frac{\xi}{R+\tilde{d}} + \frac{2}{\cos^2\delta}\tan^{-1}\frac{\eta(X+q\cos\delta) + X(R+X)\sin\delta}{\xi(R+X)\cos\delta} \qquad \left(I_4 = \frac{\xi\tilde{y}}{2(R+\tilde{d})^2}\quad \text{if } \cos\delta = 0\right)$$

By differentiation with x-coordinate (refer Appendix “Table of Differentiation of Integrals”)

$$\frac{\partial f_i{}^A}{\partial x} = \begin{pmatrix} -\dfrac{1-\alpha}{2}qY_{11} - \dfrac{\alpha}{2}\xi^2 qY_{32} \\ -\dfrac{\alpha}{2}\dfrac{\xi q}{R^3} \\ \dfrac{1-\alpha}{2}\xi Y_{11} + \dfrac{\alpha}{2}\xi q^2 Y_{32} \end{pmatrix} \qquad \frac{\partial f_i{}^B}{\partial x} = \begin{pmatrix} \xi^2 qY_{32} & -\dfrac{1-\alpha}{\alpha}J_1^x\sin\delta \\ \dfrac{\xi q}{R^3} & -\dfrac{1-\alpha}{\alpha}J_2^x\sin\delta \\ -\xi q^2 Y_{32} & -\dfrac{1-\alpha}{\alpha}J_3^x\sin\delta \end{pmatrix}$$
$$J_1^x = \frac{\partial I_1}{\partial x} \qquad J_2^x = \frac{\partial}{\partial x}\left(-\frac{\tilde{y}}{R+\tilde{d}}\right) \qquad J_3^x = \frac{\partial I_2}{\partial x}$$

$$\frac{\partial f_i{}^C}{\partial x} = \begin{pmatrix} (1-\alpha)Y_0\cos\delta & -\alpha q Z_0 \\ -(1-\alpha)\xi\left(\dfrac{\cos\delta}{R^3} + 2qY_{32}\sin\delta\right) & +\alpha\dfrac{3\tilde{c}\xi q}{R^5} \\ -(1-\alpha)\xi qY_{32}\cos\delta & +\alpha\xi\left(\dfrac{3\tilde{c}\eta}{R^5} - zY_{32} - Z_{32} - Z_0\right) \end{pmatrix}$$
$$Y_{53} = \frac{8R^2 + 9R\eta + 3\eta^2}{R^5(R+\eta)^3},\qquad Y_0 = Y_{11} - \xi^2 Y_{32}$$
$$Z_{53} = \frac{3\sin\delta}{R^5} - hY_{53},\qquad Z_0 = Z_{32} - \xi^2 Z_{53}$$

The above three vectors correspond to the contents of the row of Strike-Slip in Table 7.
(Evaluation of J_1^x et al. will be done in the later section)

(2) Dip slip

$$f_i^A = \begin{pmatrix} u_1 = \dfrac{\alpha}{2}\dfrac{q}{R} \\ u_2 = \dfrac{\Theta}{2} + \dfrac{\alpha}{2}\eta q X_{11} \\ u_3 = \dfrac{1-\alpha}{2}\ln(R+\xi) - \dfrac{\alpha}{2}q^2 X_{11} \end{pmatrix} \qquad f_i^B = \begin{pmatrix} u_1 = -\dfrac{q}{R} + \dfrac{1-\alpha}{\alpha} I_3 \sin\delta\cos\delta \\ u_2 = -\eta q X_{11} - \Theta - \dfrac{1-\alpha}{\alpha}\dfrac{\xi}{R+\tilde{d}}\sin\delta\cos\delta \\ u_3 = q^2 X_{11} + \dfrac{1-\alpha}{\alpha} I_4 \sin\delta\cos\delta \end{pmatrix}$$

$$\Theta = \tan^{-1}\frac{\xi\eta}{qR} \qquad X_{11} = \frac{1}{R(R+\xi)}$$

$$f_i^C = \begin{pmatrix} u_1 = (1-\alpha)\dfrac{\cos\delta}{R} - qY_{11}\sin\delta - \alpha\dfrac{\tilde{c}q}{R^3} \\ u_2 = (1-\alpha)\tilde{y}X_{11} - \alpha\tilde{c}\eta q X_{32} \\ u_3 = -\tilde{d}X_{11} - \xi Y_{11}\sin\delta - \alpha\tilde{c}(X_{11} - q^2 X_{32}) \end{pmatrix}$$

$$Y_{11} = \frac{1}{R(R+\eta)} \qquad X_{32} = \frac{2R+\xi}{R^3(R+\xi)^2} \qquad \tilde{c} = \tilde{d} + z$$

where, $q = y\sin\delta - d\cos\delta$, $\tilde{y} = \eta\cos\delta + q\sin\delta$, $\tilde{d} = \eta\sin\delta - q\cos\delta$, $R^2 = \xi^2 + \eta^2 + q^2 = X^2 + \eta^2$,

By differentiation with x-coordinate (refer Appendix "Table of Differentiation of Integrals")

$$\frac{\partial f_i^A}{\partial x} = \begin{pmatrix} -\dfrac{\alpha}{2}\dfrac{\xi q}{R^3} \\ -\dfrac{q}{2}Y_{11} - \dfrac{\alpha}{2}\dfrac{\eta q}{R^3} \\ \dfrac{1-\alpha}{2}\dfrac{1}{R} + \dfrac{\alpha}{2}\dfrac{q^2}{R^3} \end{pmatrix} \qquad \frac{\partial f_i^B}{\partial x} = \begin{pmatrix} \dfrac{\xi q}{R^3} + \dfrac{1-\alpha}{\alpha}J_4^x \sin\delta\cos\delta \\ \dfrac{\eta q}{R^3} + qY_{11} + \dfrac{1-\alpha}{\alpha}J_5^x \sin\delta\cos\delta \\ -\dfrac{q^2}{R^3} + \dfrac{1-\alpha}{\alpha}J_6^x \sin\delta\cos\delta \end{pmatrix}$$

$$J_4^x = \frac{\partial I_3}{\partial x} \qquad J_5^x = \frac{\partial}{\partial x}\left(-\frac{\xi}{R+\tilde{d}}\right) \qquad J_6^x = \frac{\partial I_4}{\partial x}$$

$$\frac{\partial f_i^C}{\partial x} = \begin{pmatrix} -(1-\alpha)\dfrac{\xi}{R^3}\cos\delta + \xi q Y_{32}\sin\delta + \alpha\dfrac{3\tilde{c}\xi q}{R^5} \\ -(1-\alpha)\dfrac{\tilde{y}}{R^3} + \alpha\dfrac{3\tilde{c}\eta q}{R^5} \\ \dfrac{\tilde{d}}{R^3} - Y_0\sin\delta + \alpha\dfrac{\tilde{c}}{R^3}\left(1 - \dfrac{3q^2}{R^2}\right) \end{pmatrix}$$

$$Y_{32} = \frac{2R+\eta}{R^3(R+\eta)^2} \qquad Y_0 = Y_{11} - \xi^2 Y_{32}$$

The above three vectors correspond to the contents of the row of Dip-Slip in Table 7.
(Evaluation of J_4^x et al. will be done in the later section)

(3) Tensile

$$f_i^A = \begin{pmatrix} u_1 = -\dfrac{1-\alpha}{2}\ln(R+\eta) - \dfrac{\alpha}{2}q^2 Y_{11} \\ u_2 = -\dfrac{1-\alpha}{2}\ln(R+\xi) - \dfrac{\alpha}{2}q^2 X_{11} \\ u_3 = \dfrac{\Theta}{2} - \dfrac{\alpha}{2}q(\eta X_{11} + \xi Y_{11}) \end{pmatrix} \qquad f_i^B = \begin{pmatrix} u_1 = q^2 Y_{11} - \dfrac{1-\alpha}{\alpha}I_3\sin^2\delta \\ u_2 = q^2 X_{11} + \dfrac{1-\alpha}{\alpha}\dfrac{\xi}{R+\tilde{d}}\sin^2\delta \\ u_3 = q(\eta X_{11} + \xi Y_{11}) - \Theta - \dfrac{1-\alpha}{\alpha}I_4\sin^2\delta \end{pmatrix}$$

$$\Theta = \tan^{-1}\frac{\xi\eta}{qR} \qquad X_{11} = \frac{1}{R(R+\xi)} \qquad Y_{11} = \frac{1}{R(R+\eta)}$$

$$f_i^C = \begin{pmatrix} u_1 = -(1-\alpha)\left(\dfrac{\sin\delta}{R} + qY_{11}\cos\delta\right) - \alpha(zY_{11} - q^2 Z_{32}) \\ u_2 = (1-\alpha)2\xi Y_{11}\sin\delta + \tilde{d}X_{11} - \alpha\tilde{c}(X_{11} - q^2 X_{32}) \\ u_3 = (1-\alpha)(\tilde{y}X_{11} + \xi Y_{11}\cos\delta) + \alpha q(\tilde{c}\eta X_{32} + \xi Z_{32}) \end{pmatrix}$$

$$X_{32} = \frac{2R+\xi}{R^3(R+\xi)^2}, \quad Y_{32} = \frac{2R+\eta}{R^3(R+\eta)^2}$$

$$Z_{32} = \frac{\sin\delta}{R^3} - hY_{32}$$

$$h = q\cos\delta - z, \quad \tilde{c} = \tilde{d} + z = \eta\sin\delta - h$$

where, $q = y\sin\delta - d\cos\delta$, $\tilde{y} = \eta\cos\delta + q\sin\delta$, $\tilde{d} = \eta\sin\delta - q\cos\delta$, $R^2 = \xi^2 + \eta^2 + q^2 = X^2 + \eta^2$,

By differentiation with x-coordinate (refer Appendix "Table of Differentiation of Integrals")

$$\frac{\partial f_i^A}{\partial x} = \begin{pmatrix} -\dfrac{1-\alpha}{2}\xi Y_{11} + \dfrac{\alpha}{2}\xi q^2 Y_{32} \\ -\dfrac{1-\alpha}{2}\dfrac{1}{R} + \dfrac{\alpha}{2}\dfrac{q^2}{R^3} \\ -\dfrac{1-\alpha}{2}qY_{11} - \dfrac{\alpha}{2}q^3 Y_{32} \end{pmatrix} \qquad \frac{\partial f_i^B}{\partial x} = \begin{pmatrix} -\xi q^2 Y_{32} - \dfrac{1-\alpha}{\alpha}J_4^x\sin^2\delta \\ -\dfrac{q^2}{R^3} - \dfrac{1-\alpha}{\alpha}J_5^x\sin^2\delta \\ q^3 Y_{32} - \dfrac{1-\alpha}{\alpha}J_6^x\sin^2\delta \end{pmatrix}$$

$$J_4^x = \frac{\partial I_3}{\partial x} \qquad J_5^x = \frac{\partial}{\partial x}\left(-\frac{\xi}{R+\tilde{d}}\right) \qquad J_6^x = \frac{\partial I_4}{\partial x}$$

$$\frac{\partial f_i^C}{\partial x} = \begin{pmatrix} (1-\alpha)\left(\dfrac{\xi}{R^3}\sin\delta + \xi q Y_{32}\cos\delta\right) + \alpha\xi(zY_{32} - q^2 Z_{53}) \\ 2(1-\alpha)Y_0\sin\delta - \dfrac{\tilde{d}}{R^3} + \alpha\dfrac{\tilde{c}}{R^3}\left(1 - \dfrac{3q^2}{R^2}\right) \\ -(1-\alpha)\left(\dfrac{\tilde{y}}{R^3} - Y_0\cos\delta\right) - \alpha\left(\dfrac{3\tilde{c}\eta q}{R^5} - qZ_0\right) \end{pmatrix}$$

$$Y_{53} = \frac{8R^2 + 9R\eta + 3\eta^2}{R^5(R+\eta)^3}, \quad Y_0 = Y_{11} - \xi^2 Y_{32}$$

$$Z_{53} = \frac{3\sin\delta}{R^5} - hY_{53}, \quad Z_0 = Z_{32} - \xi^2 Z_{53}$$

$$h = q\cos\delta - z, \quad \tilde{c} = \tilde{d} + z = \eta\sin\delta - h$$

From (∗ 6) of Appendix, $q^2 Z_{53} = -\frac{3\tilde{c}\eta}{R^5} - hY_{32} + 2Z_{32} - Z_0$

Therefore, $\frac{\partial f_1{}^C}{\partial x} = (1-\alpha)\left(\frac{\xi}{R^3}\sin\delta + \xi q Y_{32}\cos\delta\right) + \alpha\xi\left[\frac{3\tilde{c}\eta}{R^5} + (h+z)Y_{32} - 2Z_{32} + Z_0\right]$

$$= (1-\alpha)\frac{\xi}{R^3}\sin\delta + \xi q Y_{32}\cos\delta + \alpha\xi\left(\frac{3\tilde{c}\eta}{R^5} - 2Z_{32} + Z_0\right)$$

The above three vectors correspond to the contents of the row of Tensile in Table 7.
(Evaluation of J_4^x et al. will be done in the later section)

[II] Derivation of Table 8 (y-Derivative)

Table 8 can be derived by differentiation of Table 6 with y-coordinate.
In the following, the notation is matched with Tables in Okada (1992).

Displacement : $u_x(x,y,z) = \frac{U}{2\pi}\left[\ u_1{}^A - \hat{u}_1{}^A + u_1{}^B + zu_1{}^C\ \right]$

$$u_y(x,y,z) = \frac{U}{2\pi}\left[(u_2{}^A - \hat{u}_2{}^A + u_2{}^B + zu_2{}^C)\cos\delta - (u_3{}^A - \hat{u}_3{}^A + u_3{}^B + zu_3{}^C)\sin\delta\,\right]$$

$$u_z(x,y,z) = \frac{U}{2\pi}\left[(u_2{}^A - \hat{u}_2{}^A + u_2{}^B - zu_2{}^C)\sin\delta + (u_3{}^A - \hat{u}_3{}^A + u_3{}^B - zu_3{}^C)\cos\delta\,\right]$$

$$u_i{}^A = f_i{}^A(\xi,\eta,z)|_{\xi=x}^{\xi=x-L}\cdot|_{\eta=p}^{\eta=p-W}, \quad \hat{u}_i{}^A = f_i{}^A(\xi,\eta,-z)\|, \quad u_i{}^B = f_i{}^B(\xi,\eta,z)\|, \quad u_i{}^C = f_i{}^C(\xi,\eta,z)\|$$

y-Derivative : $\frac{\partial u_x}{\partial y}(x,y,z) = \frac{U}{2\pi}\left[\,k_1{}^A - \hat{k}_1{}^A + k_1{}^B + zk_1{}^C\,\right]$

$$\frac{\partial u_y}{\partial y}(x,y,z) = \frac{U}{2\pi}\left[(k_2{}^A - \hat{k}_2{}^A + k_2{}^B + zk_2{}^C)\cos\delta - (k_3{}^A - \hat{k}_3{}^A + k_3{}^B + zk_3{}^C)\sin\delta\,\right]$$

$$\frac{\partial u_z}{\partial y}(x,y,z) = \frac{U}{2\pi}\left[(k_2{}^A - \hat{k}_2{}^A + k_2{}^B - zk_2{}^C)\sin\delta + (k_3{}^A - \hat{k}_3{}^A + k_3{}^B - zk_3{}^C)\cos\delta\,\right]$$

$$k_i{}^A = \partial f_i{}^A/\partial y(\xi,\eta,z)|_{\xi=x}^{\xi=x-L}\cdot|_{\eta=p}^{\eta=p-W}, \quad \hat{k}_i{}^A = \partial f_i{}^A/\partial y(\xi,\eta,-z)\|, \quad k_i{}^B = \partial f_i{}^B/\partial y(\xi,\eta,z)\|, \quad k_i{}^C = \partial f_i{}^C/\partial y(\xi,\eta,z)\|$$

(1) Strike slip

$$f_i{}^A = \begin{pmatrix} u_1 = \frac{\Theta}{2} & +\frac{\alpha}{2}\xi q Y_{11} \\ u_2 = & \frac{\alpha}{2}\frac{q}{R} \\ u_3 = \frac{1-\alpha}{2}\ln(R+\eta) & -\frac{\alpha}{2}q^2 Y_{11} \end{pmatrix} \qquad f_i{}^B = \begin{pmatrix} u_1 = -\xi q Y_{11} - \Theta & -\frac{1-\alpha}{\alpha}I_1\sin\delta \\ u_2 = -\frac{q}{R} & +\frac{1-\alpha}{\alpha}\frac{\tilde{y}}{R+\tilde{d}}\sin\delta \\ u_3 = q^2 Y_{11} & -\frac{1-\alpha}{\alpha}I_2\sin\delta \end{pmatrix}$$

$$\Theta = \tan^{-1}\frac{\xi\eta}{qR} \qquad Y_{11} = \frac{1}{R(R+\eta)}$$

$$f_i{}^C = \begin{pmatrix} u_1 = (1-\alpha)\xi Y_{11}\cos\delta & -\alpha\xi q Z_{32} \\ u_2 = (1-\alpha)\left(\frac{\cos\delta}{R} + 2qY_{11}\sin\delta\right) - \alpha\frac{\tilde{c}q}{R^3} \\ u_3 = (1-\alpha)qY_{11}\cos\delta - \alpha\left(\frac{\tilde{c}\eta}{R^3} - zY_{11} + \xi^2 Z_{32}\right) \end{pmatrix}$$

$$Y_{32} = \frac{2R+\eta}{R^3(R+\eta)^2} \qquad Z_{32} = \frac{\sin\delta}{R^3} - hY_{32}$$

$$h = q\cos\delta - z\,, \quad \tilde{c} = \tilde{d} + z = \eta\sin\delta - h$$

where, $q = y\sin\delta - d\cos\delta$, $\tilde{y} = \eta\cos\delta + q\sin\delta$, $\tilde{d} = \eta\sin\delta - q\cos\delta$, $R^2 = \xi^2 + \eta^2 + q^2 = X^2 + \eta^2$,

By differentiation with y-coordinate (refer Appendix "Table of Differentiation of Integrals")

$$\frac{\partial f_i{}^A}{\partial y} = \begin{pmatrix} \frac{1}{2}(\tilde{d}X_{11} + \xi Y_{11}\sin\delta) + \frac{\alpha}{2}\xi\left(\frac{\tilde{d}}{R^3} - (Y_{11} - \xi^2 Y_{32})\sin\delta\right) \\ \frac{\alpha}{2}\left(\frac{\sin\delta}{R} - \frac{\tilde{y}q}{R^3}\right) \\ \frac{1-\alpha}{2}\left(\frac{\cos\delta}{R} + qY_{11}\sin\delta\right) - \frac{\alpha}{2}q\left(\frac{\tilde{d}}{R^3} + \xi^2 Y_{32}\sin\delta\right) \end{pmatrix} = \begin{pmatrix} \frac{1-\alpha}{2}\xi Y_{11}\sin\delta + \frac{\tilde{d}}{2}X_{11} & +\frac{\alpha}{2}\xi F \\ & \frac{\alpha}{2}E \\ \frac{1-\alpha}{2}\left(\frac{\cos\delta}{R} + qY_{11}\sin\delta\right) - \frac{\alpha}{2}qF \end{pmatrix}$$

$$E = \frac{\sin\delta}{R} - \frac{\tilde{y}q}{R^3} \qquad F = \frac{\tilde{d}}{R^3} + \xi^2 Y_{32}\sin\delta$$

$$\frac{\partial f_i{}^B}{\partial y} = \begin{pmatrix} -\xi\left(\frac{\tilde{d}}{R^3} - (Y_{11} - \xi^2 Y_{32})\sin\delta\right) - (\tilde{d}X_{11} + \xi Y_{11}\sin\delta) + \frac{1-\alpha}{\alpha}J_1^y\sin\delta \\ -\left(\frac{\sin\delta}{R} - \frac{\tilde{y}q}{R^3}\right) \qquad + \frac{1-\alpha}{\alpha}J_2^y\sin\delta \\ q\left(\frac{\tilde{d}}{R^3} + \xi^2 Y_{32}\sin\delta\right) \qquad - \frac{1-\alpha}{\alpha}J_3^y\sin\delta \end{pmatrix} = \begin{pmatrix} -\xi F - \tilde{d}X_{11} + \frac{1-\alpha}{\alpha}J_1^y\sin\delta \\ -E \qquad + \frac{1-\alpha}{\alpha}J_2^y\sin\delta \\ qF \qquad - \frac{1-\alpha}{\alpha}J_3^y\sin\delta \end{pmatrix}$$

$$J_1^y = -\frac{\partial I_1}{\partial y} \qquad J_2^y = \frac{\partial}{\partial y}\left(\frac{\tilde{y}}{R+\tilde{d}}\right) \qquad J_3^y = \frac{\partial I_2}{\partial y}$$

$$\frac{\partial f_i^{\,C}}{\partial y}=\begin{pmatrix} -(1-\alpha)\xi\left(\frac{\cos\delta}{R^3}+qY_{32}\sin\delta\right)\cos\delta & -\alpha\xi\left[\frac{3\tilde{c}\tilde{d}}{R^5}-(zY_{32}+Z_{32}+Z_0)\sin\delta\right] \\ (1-\alpha)\left\{-\frac{\tilde{y}}{R^3}\cos\delta+2\left(\frac{\tilde{d}}{R^3}-Y_0\sin\delta\right)\sin\delta\right\} & -\alpha\tilde{c}\left(\frac{\sin\delta}{R^3}-\frac{3\tilde{y}q}{R^5}\right) \\ \multicolumn{2}{l}{(1-\alpha)\left(\frac{\tilde{d}}{R^3}-Y_0\sin\delta\right)\cos\delta-\alpha\left[\tilde{c}\left(\frac{\cos\delta}{R^3}-\frac{3\tilde{y}\eta}{R^5}\right)+z\left(\frac{\cos\delta}{R^3}+qY_{32}\sin\delta\right)-\xi^2\left(\frac{3\tilde{c}}{R^5}\cos\delta+(Y_{32}\cos\delta+qZ_{53})\sin\delta\right)\right]} \end{pmatrix}$$

Since $\tilde{y}\cos\delta+\tilde{d}\sin\delta=\eta$,

$$\frac{\partial f_2^{\,C}}{\partial y}=2(1-\alpha)\left(\frac{\tilde{d}}{R^3}-Y_0\sin\delta\right)\sin\delta-\frac{\tilde{y}}{R^3}\cos\delta-\alpha\left(-\frac{\tilde{y}}{R^3}\cos\delta+\frac{\tilde{c}}{R^3}\sin\delta-\frac{3\tilde{c}\tilde{y}q}{R^5}\right)$$
$$=2(1-\alpha)\left(\frac{\tilde{d}}{R^3}-Y_0\sin\delta\right)\sin\delta-\frac{\tilde{y}}{R^3}\cos\delta-\alpha\left(\frac{\tilde{c}+\tilde{d}}{R^3}\sin\delta-\frac{\eta}{R^3}-\frac{3\tilde{c}\tilde{y}q}{R^5}\right)$$

Since $\tilde{y}\sin\delta-\tilde{d}\cos\delta=q$, $z=\tilde{c}-\tilde{d}$ and $(Y_{11}-\xi^2Y_{32})+(Y_{11}-q^2Y_{32})=\frac{\eta}{R^3}$ (refer (∗ 3) of Appendix)

$$\frac{\partial f_3^{\,C}}{\partial y}=-(1-\alpha)\frac{q}{R^3}+(1-\alpha)\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\sin\delta-\alpha\left\{\frac{\tilde{c}+z}{R^3}\cos\delta-\frac{3\tilde{c}(\tilde{y}\eta+\xi^2\cos\delta)}{R^5}+(q\cos\delta-h)qY_{32}\sin\delta-\xi^2(Y_{32}\cos\delta+qZ_{53})\sin\delta\right\}$$
$$=-(1-\alpha)\frac{q}{R^3}+\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\sin\delta-\alpha\left\{\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\sin\delta+\frac{\tilde{c}+z}{R^3}\cos\delta-\frac{3\tilde{c}(\tilde{d}q+R^2\cos\delta)}{R^5}+(q^2-\xi^2)Y_{32}\sin\delta\cos\delta-qhY_{32}\sin\delta-\xi^2qZ_{53}\sin\delta\right\}$$
$$=-(1-\alpha)\frac{q}{R^3}+\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\sin\delta-\alpha\left\{\frac{\eta}{R^3}\sin\delta\cos\delta+\frac{q^2}{R^3}\sin^2\delta+\frac{z-2\tilde{c}}{R^3}\cos\delta-\frac{3\tilde{c}\tilde{d}q}{R^5}+[(q^2-\xi^2)Y_{32}-Y_0]\sin\delta\cos\delta-q\sin\delta\,(hY_{32}+\xi^2Z_{53})\right\}$$
$$=-(1-\alpha)\frac{q}{R^3}+\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\sin\delta-\alpha\left\{\frac{\eta\sin\delta+z-2\tilde{c}}{R^3}\cos\delta-\frac{3\tilde{c}\tilde{d}q}{R^5}-[Y_{11}-q^2Y_{32}]\sin\delta\cos\delta+q\sin\delta\left(\frac{\sin\delta}{R^3}-hY_{32}-\xi^2Z_{53}\right)\right\}$$
$$=-(1-\alpha)\frac{q}{R^3}+\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\sin\delta-\alpha\left\{\frac{\eta\sin\delta-\tilde{c}-\tilde{d}}{R^3}\cos\delta-\frac{3\tilde{c}\tilde{d}q}{R^5}-\left[\frac{\eta}{R^3}-Y_0\right]\sin\delta\cos\delta+qZ_0\sin\delta\right\}$$
$$=-(1-\alpha)\frac{q}{R^3}+\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\sin\delta+\alpha\left\{\frac{\tilde{c}+\tilde{d}}{R^3}\cos\delta+\frac{3\tilde{c}\tilde{d}q}{R^5}-(Y_0\cos\delta+qZ_0)\sin\delta\right\}$$

The above three vectors correspond to the contents of the row of Strike-Slip in Table 8.
(Evaluation of J_1^y et al. will be done in the later section)

(2) Dip slip

$$f_i^{\,A}=\begin{pmatrix} u_1= & \frac{\alpha}{2}\frac{q}{R} \\ u_2=\frac{\Theta}{2} & +\frac{\alpha}{2}\eta qX_{11} \\ u_3=\frac{1-\alpha}{2}\ln(R+\xi) & -\frac{\alpha}{2}q^2X_{11} \end{pmatrix}\qquad f_i^{\,B}=\begin{pmatrix} u_1=-\frac{q}{R} & +\frac{1-\alpha}{\alpha}I_3\sin\delta\cos\delta \\ u_2=-\eta qX_{11}-\Theta & -\frac{1-\alpha}{\alpha}\frac{\xi}{R+\tilde{d}}\sin\delta\cos\delta \\ u_3=q^2X_{11} & +\frac{1-\alpha}{\alpha}I_4\sin\delta\cos\delta \end{pmatrix}\qquad \begin{array}{l}\Theta=\tan^{-1}\frac{\xi\eta}{qR}\\ X_{11}=\frac{1}{R(R+\xi)}\end{array}$$

$$f_i^{\,C}=\begin{pmatrix} u_1= & (1-\alpha)\frac{\cos\delta}{R}-qY_{11}\sin\delta & -\alpha\frac{\tilde{c}q}{R^3} \\ u_2= & (1-\alpha)\tilde{y}X_{11} & -\alpha\tilde{c}\eta qX_{32} \\ u_3= & -\tilde{d}X_{11}-\xi Y_{11}\sin\delta & -\alpha\tilde{c}(X_{11}-q^2X_{32}) \end{pmatrix}\qquad \begin{array}{l} Y_{11}=\frac{1}{R(R+\eta)}\\ X_{32}=\frac{2R+\xi}{R^3(R+\xi)^2}\\ \tilde{c}=\tilde{d}+z\end{array}$$

where, $q=y\sin\delta-d\cos\delta$, $\tilde{y}=\eta\cos\delta+q\sin\delta$, $\tilde{d}=\eta\sin\delta-q\cos\delta$, $R^2=\xi^2+\eta^2+q^2=X^2+\eta^2$,

By differentiation with y-coordinate (refer Appendix "Table of Differentiation of Integrals")

$$\frac{\partial f_i^{\,A}}{\partial y}=\begin{pmatrix} & \frac{\alpha}{2}\left(\frac{\sin\delta}{R}-\frac{\tilde{y}q}{R^3}\right) \\ \frac{1}{2}(\tilde{d}X_{11}+\xi Y_{11}\sin\delta) & +\frac{\alpha}{2}[(2\eta\sin\delta-\tilde{d})X_{11}-\tilde{y}\eta qX_{32}] \\ \frac{1-\alpha}{2}\tilde{y}X_{11} & -\frac{\alpha}{2}q(2X_{11}\sin\delta-\tilde{y}qX_{32}) \end{pmatrix}=\begin{pmatrix} & \frac{\alpha}{2}E \\ \frac{1-\alpha}{2}\tilde{d}X_{11}+\frac{\xi}{2}Y_{11}\sin\delta & +\frac{\alpha}{2}\eta G \\ \frac{1-\alpha}{2}\tilde{y}X_{11} & -\frac{\alpha}{2}qG \end{pmatrix}\qquad \begin{array}{l} E=\frac{\sin\delta}{R}-\frac{\tilde{y}q}{R^3}\\ G=2X_{11}\sin\delta-\tilde{y}qX_{32}\end{array}$$

$$\frac{\partial f_i^{\,B}}{\partial y}=\begin{pmatrix} -\left(\frac{\sin\delta}{R}-\frac{\tilde{y}q}{R^3}\right) & +\frac{1-\alpha}{\alpha}J_4^y\sin\delta\cos\delta \\ -[(2\eta\sin\delta-\tilde{d})X_{11}-\tilde{y}\eta qX_{32}]-(\tilde{d}X_{11}+\xi Y_{11}\sin\delta) & +\frac{1-\alpha}{\alpha}J_5^y\sin\delta\cos\delta \\ q(2X_{11}\sin\delta-\tilde{y}qX_{32}) & +\frac{1-\alpha}{\alpha}J_6^y\sin\delta\cos\delta \end{pmatrix}$$

$$=\begin{pmatrix} -E & +\frac{1-\alpha}{\alpha}J_4^y\sin\delta\cos\delta \\ -\eta G-\xi Y_{11}\sin\delta & +\frac{1-\alpha}{\alpha}J_5^y\sin\delta\cos\delta \\ qG & +\frac{1-\alpha}{\alpha}J_6^y\sin\delta\cos\delta \end{pmatrix}\qquad \begin{array}{l} J_4^y=\frac{\partial I_3}{\partial y}\\ J_5^y=\frac{\partial}{\partial y}\left(-\frac{\xi}{R+\tilde{d}}\right)\\ J_6^y=\frac{\partial I_4}{\partial y}\end{array}$$

$$\frac{\partial f_i^{\,C}}{\partial y}=\begin{pmatrix} -(1-\alpha)\dfrac{\tilde y}{R^3}\cos\delta-\left(\dfrac{\tilde d}{R^3}-Y_0\sin\delta\right)\sin\delta-\alpha\tilde c\left(\dfrac{\sin\delta}{R^3}-\dfrac{3\tilde y q}{R^5}\right)\\ (1-\alpha)(X_{11}-\tilde y^2X_{32}) \qquad -\alpha\tilde c[(\tilde d+2q\cos\delta)X_{32}-\tilde y\eta qX_{53}]\\ \tilde y\tilde dX_{32}+\xi\left(\dfrac{\cos\delta}{R^3}+qY_{32}\sin\delta\right)\sin\delta \qquad +\alpha\tilde c[\tilde yX_{32}+q(2X_{32}\sin\delta-\tilde yqX_{53})]\end{pmatrix}$$

Since $\tilde y\cos\delta+\tilde d\sin\delta=\eta$, $\dfrac{\partial f_1^{\,C}}{\partial y}=-(1-\alpha)\left(\dfrac{\eta}{R^3}-\dfrac{\tilde d}{R^3}\sin\delta\right)-\dfrac{\tilde d}{R^3}\sin\delta+Y_0\sin^2\delta-\alpha\left(\dfrac{\tilde c}{R^3}\sin\delta-\dfrac{3\tilde c\tilde yq}{R^5}\right)$

$$=-(1-\alpha)\frac{\eta}{R^3}+Y_0\sin^2\delta-\alpha\left(\frac{\tilde c+\tilde d}{R^3}\sin\delta-\frac{3\tilde c\tilde yq}{R^5}\right)$$

The above three vectors correspond to the contents of the row of Dip-Slip in Table 8.
(Evaluation of J_4^y et al. will be done in the later section)

(3) Tensile

$$f_i^{\,A}=\begin{pmatrix} u_1=-\dfrac{1-\alpha}{2}\ln(R+\eta)-\dfrac{\alpha}{2}q^2Y_{11}\\ u_2=-\dfrac{1-\alpha}{2}\ln(R+\xi)-\dfrac{\alpha}{2}q^2X_{11}\\ u_3=\dfrac{\Theta}{2}\qquad-\dfrac{\alpha}{2}q(\eta X_{11}+\xi Y_{11})\end{pmatrix}\qquad f_i^{\,B}=\begin{pmatrix} u_1=q^2Y_{11} \qquad -\dfrac{1-\alpha}{\alpha}I_3\sin^2\delta\\ u_2=q^2X_{11} \qquad +\dfrac{1-\alpha}{\alpha}\dfrac{\xi}{R+\tilde d}\sin^2\delta\\ u_3=q(\eta X_{11}+\xi Y_{11})-\Theta-\dfrac{1-\alpha}{\alpha}I_4\sin^2\delta\end{pmatrix}$$

$$\Theta=\tan^{-1}\frac{\xi\eta}{qR}\qquad X_{11}=\frac{1}{R(R+\xi)}\qquad Y_{11}=\frac{1}{R(R+\eta)}$$

$$f_i^{\,C}=\begin{pmatrix} u_1=-(1-\alpha)\left(\dfrac{\sin\delta}{R}+qY_{11}\cos\delta\right)-\alpha(zY_{11}-q^2Z_{32})\\ u_2=(1-\alpha)2\xi Y_{11}\sin\delta+\tilde dX_{11}\quad-\alpha\tilde c(X_{11}-q^2X_{32})\\ u_3=(1-\alpha)(\tilde yX_{11}+\xi Y_{11}\cos\delta)\ +\alpha q(\tilde c\eta X_{32}+\xi Z_{32})\end{pmatrix}$$

$$X_{32}=\frac{2R+\xi}{R^3(R+\xi)^2}\ ,\quad Y_{32}=\frac{2R+\eta}{R^3(R+\eta)^2}\qquad Z_{32}=\frac{\sin\delta}{R^3}-hY_{32}$$

$$h=q\cos\delta-z\ ,\qquad \tilde c=\tilde d+z=\eta\sin\delta-h$$

where, $q=y\sin\delta-d\cos\delta$, $\tilde y=\eta\cos\delta+q\sin\delta$, $\tilde d=\eta\sin\delta-q\cos\delta$, $R^2=\xi^2+\eta^2+q^2=X^2+\eta^2$,

By differentiation with y-coordinate (refer Appendix "Table of Differentiation of Integrals")

$$\frac{\partial f_i^{\,A}}{\partial y}=\begin{pmatrix} -\dfrac{1-\alpha}{2}\left(\dfrac{\cos\delta}{R}+qY_{11}\sin\delta\right)-\dfrac{\alpha}{2}q\left(\dfrac{\tilde d}{R^3}+\xi^2Y_{32}\sin\delta\right)\\ -\dfrac{1-\alpha}{2}\tilde yX_{11}\qquad-\dfrac{\alpha}{2}q(2X_{11}\sin\delta-\tilde yqX_{32})\\ \dfrac{1}{2}(\tilde dX_{11}+\xi Y_{11}\sin\delta)\ -\dfrac{\alpha}{2}\left\{(\tilde d+2q\cos\delta)X_{11}-\tilde y\eta qX_{32}+\xi\left(\dfrac{\tilde d}{R^3}-Y_0\sin\delta\right)\right\}\end{pmatrix}\qquad Y_0=Y_{11}-\xi^2Y_{32}$$

Since $(Y_{11}-\xi^2Y_{32})+(Y_{11}-q^2Y_{32})=\dfrac{\eta}{R^3}$ (refer (＊3) of Appendix)

$$\frac{\partial f_3^{\,A}}{\partial y}=\frac{1}{2}(\tilde dX_{11}+\xi Y_{11}\sin\delta)-\frac{\alpha}{2}\left\{\tilde dX_{11}+2qX_{11}\cos\delta-\tilde y\eta qX_{32}+\xi\left(\frac{\tilde d}{R^3}-\left[\frac{\eta}{R^3}-(Y_{11}-q^2Y_{32})\right]\sin\delta\right)\right\}$$
$$=\frac{1-\alpha}{2}(\tilde dX_{11}+\xi Y_{11}\sin\delta)-\frac{\alpha}{2}\left\{2qX_{11}\cos\delta-\tilde y\eta qX_{32}-\frac{\xi q}{R^3}\cos\delta-\xi q^2Y_{32}\sin\delta\right\}$$
$$=\frac{1-\alpha}{2}(\tilde dX_{11}+\xi Y_{11}\sin\delta)+\frac{\alpha}{2}qH$$

Here, $H=-2X_{11}\cos\delta+\tilde y\eta X_{32}+\dfrac{\xi}{R^3}\cos\delta+\xi qY_{32}\sin\delta$

$$=-2X_{11}\cos\delta+(\eta\cos\delta+q\sin\delta)\eta X_{32}+\frac{\xi}{R^3}\cos\delta+\xi qY_{32}\sin\delta$$
$$=\left(\eta^2X_{32}+\frac{\xi}{R^3}-2X_{11}\right)\cos\delta+\eta qX_{32}\sin\delta+\xi qY_{32}\sin\delta$$
$$=\frac{2R+\xi}{R^3(R+\xi)^2}(\xi^2+\eta^2-R^2)\cos\delta+\eta qX_{32}\sin\delta+\xi qY_{32}\sin\delta$$
$$=-q^2X_{32}\cos\delta+\eta qX_{32}\sin\delta+\xi qY_{32}\sin\delta$$
$$=(\eta\sin\delta-q\cos\delta)qX_{32}+\xi qY_{32}\sin\delta=\tilde dqX_{32}+\xi qY_{32}\sin\delta$$

Therefore,

$$\frac{\partial f_i^{\,A}}{\partial y}=\begin{pmatrix} -\dfrac{1-\alpha}{2}\left(\dfrac{\cos\delta}{R}+qY_{11}\sin\delta\right)-\dfrac{\alpha}{2}qF\\ -\dfrac{1-\alpha}{2}\tilde yX_{11}\qquad-\dfrac{\alpha}{2}qG\\ \dfrac{1-\alpha}{2}(\tilde dX_{11}+\xi Y_{11}\sin\delta)\ +\dfrac{\alpha}{2}qH\end{pmatrix}\qquad \begin{aligned}F&=\frac{\tilde d}{R^3}+\xi^2Y_{32}\sin\delta\\ G&=2X_{11}\sin\delta-\tilde yqX_{32}\\ H&=\tilde dqX_{32}+\xi qY_{32}\sin\delta\end{aligned}$$

$$\frac{\partial f_i^{\,B}}{\partial y}=\begin{pmatrix} q\left(\dfrac{\tilde d}{R^3}+\xi^2 Y_{32}\sin\delta\right) & -\dfrac{1-\alpha}{\alpha}J_4^y\sin^2\delta \\ q(2X_{11}\sin\delta-\tilde y qX_{32}) & -\dfrac{1-\alpha}{\alpha}J_5^y\sin^2\delta \\ (\tilde d+2q\cos\delta)X_{11}-\tilde y\eta qX_{32}+\xi\left(\dfrac{\tilde d}{R^3}-Y_0\sin\delta\right)-(\tilde dX_{11}+\xi Y_{11}\sin\delta) & -\dfrac{1-\alpha}{\alpha}J_6^y\sin^2\delta \end{pmatrix}\qquad \begin{aligned} J_4^y&=\frac{\partial I_3}{\partial y}\\ J_5^y&=\frac{\partial}{\partial y}\left(-\frac{\xi}{R+\tilde d}\right)\\ J_6^y&=\frac{\partial I_4}{\partial y}\end{aligned}$$

$$=\begin{pmatrix} q\left(\dfrac{\tilde d}{R^3}+\xi^2 Y_{32}\sin\delta\right) & -\dfrac{1-\alpha}{\alpha}J_4^y\sin^2\delta \\ q(2X_{11}\sin\delta-\tilde y qX_{32}) & -\dfrac{1-\alpha}{\alpha}J_5^y\sin^2\delta \\ 2qX_{11}\cos\delta-\tilde y\eta qX_{32}-\dfrac{\xi q}{R^3}\cos\delta-\xi q^2Y_{32}\sin\delta & -\dfrac{1-\alpha}{\alpha}J_6^y\sin^2\delta \end{pmatrix}=\begin{pmatrix} qF-\dfrac{1-\alpha}{\alpha}J_4^y\sin^2\delta \\ qG-\dfrac{1-\alpha}{\alpha}J_5^y\sin^2\delta \\ -qH-\dfrac{1-\alpha}{\alpha}J_6^y\sin^2\delta \end{pmatrix}$$

$$\frac{\partial f_i^{\,C}}{\partial y}=\begin{pmatrix} (1-\alpha)\left\{\dfrac{\tilde y}{R^3}\sin\delta-\left(\dfrac{\tilde d}{R^3}-Y_0\right)\sin\delta\cos\delta\right\} & +\alpha\left\{z\left(\dfrac{\cos\delta}{R^3}+qY_{32}\sin\delta\right)+q\left[\dfrac{3\tilde c\tilde d}{R^5}-(zY_{32}+Z_0)\sin\delta\right]\right\} \\ -2(1-\alpha)\xi\left(\dfrac{\cos\delta}{R^3}+qY_{32}\sin\delta\right)\sin\delta-\tilde y\tilde dX_{32} & +\alpha\tilde c\;[\tilde yX_{32}+(2qX_{32}\sin\delta-\tilde yq^2X_{53})] \\ (1-\alpha)\left\{X_{11}-\tilde y^2X_{32}-\xi\left(\dfrac{\cos\delta}{R^3}+qY_{32}\sin\delta\right)\cos\delta\right\} & +\alpha\left\{\tilde c[(\tilde d+2q\cos\delta)X_{32}-\tilde y\eta qX_{53}]+\xi\left[\dfrac{3\tilde c\tilde d}{R^5}-(zY_{32}+Z_{32}+Z_0)\sin\delta\right]\right\} \end{pmatrix}$$

$$=\begin{pmatrix} (1-\alpha)\left(\dfrac{q}{R^3}+Y_0\sin\delta\cos\delta\right) & +\alpha\left(\dfrac{z}{R^3}\cos\delta+\dfrac{3\tilde c\tilde dq}{R^5}-q\,Z_0\sin\delta\right) \\ -2(1-\alpha)\xi P\sin\delta-\tilde y\tilde dX_{32} & +\alpha\tilde c\;[(\tilde y+2q\sin\delta)X_{32}-\tilde yq^2X_{53}] \\ -(1-\alpha)(\xi P\cos\delta-X_{11}+\tilde y^2X_{32}) & +\alpha\tilde c[(\tilde d+2q\cos\delta)X_{32}-\tilde y\eta qX_{53}] \end{pmatrix}\qquad \begin{aligned} P&=\frac{\cos\delta}{R^3}+qY_{32}\sin\delta\\ Q&=\frac{3\tilde c\tilde d}{R^5}-(zY_{32}+Z_{32}+Z_0)\sin\delta\end{aligned}$$

The above three vectors correspond to the contents of the row of Tensile in Table 7.
(Evaluation of J_4^y et al. will be done in the later section)

[III] Derivation of Table 9 (z-Derivative)

Table 9 can be derived by differentiation of Table 6 with z-coordinate.
In the following, the notation is matched with Tables in Okada (1992).

Displacement :

$$u_x(x,y,z)=\frac{U}{2\pi}\left[\,u_1^{\,A}-\hat u_1^{\,A}+u_1^{\,B}+zu_1^{\,C}\,\right]$$

$$u_y(x,y,z)=\frac{U}{2\pi}\left[(u_2^{\,A}-\hat u_2^{\,A}+u_2^{\,B}+zu_2^{\,C})\cos\delta-(u_3^{\,A}-\hat u_3^{\,A}+u_3^{\,B}+zu_3^{\,C})\sin\delta\,\right]$$

$$u_z(x,y,z)=\frac{U}{2\pi}\left[(u_2^{\,A}-\hat u_2^{\,A}+u_2^{\,B}-zu_2^{\,C})\sin\delta+(u_3^{\,A}-\hat u_3^{\,A}+u_3^{\,B}-zu_3^{\,C})\cos\delta\,\right]$$

$$u_i^{\,A}=f_i^{\,A}(\xi,\eta,z)\Big|_{\xi=x}^{\xi=x-L}.\Big|_{\eta=p}^{\eta=p-W},\quad \hat u_i^{\,A}=f_i^{\,A}(\xi,\eta,-z)\|,\quad u_i^{\,B}=f_i^{\,B}(\xi,\eta,z)\|,\quad u_i^{\,C}=f_i^{\,C}(\xi,\eta,z)\|$$

z-Derivative :

$$\frac{\partial u_x}{\partial z}(x,y,z)=\frac{U}{2\pi}\left[\,l_1^{\,A}+\hat l_1^{\,A}+l_1^{\,B}+u_1^{\,C}+zl_1^{\,C}\,\right]$$

$$\frac{\partial u_y}{\partial z}(x,y,z)=\frac{U}{2\pi}\left[(l_2^{\,A}+\hat l_2^{\,A}+l_2^{\,B}+u_2^{\,C}+zl_2^{\,C})\cos\delta-(l_3^{\,A}+\hat l_3^{\,A}+l_3^{\,B}+u_3^{\,C}+zl_3^{\,C})\sin\delta\,\right]$$

$$\frac{\partial u_z}{\partial z}(x,y,z)=\frac{U}{2\pi}\left[(l_2^{\,A}+\hat l_2^{\,A}+l_2^{\,B}-u_2^{\,C}-zl_2^{\,C})\sin\delta+(l_3^{\,A}+\hat l_3^{\,A}+l_3^{\,B}-+u_3^{\,C}-zl_3^{\,C})\cos\delta\,\right]$$

$$l_i^{\,A}=\partial f_i^{\,A}/\partial z(\xi,\eta,z)\Big|_{\xi=x}^{\xi=x-L}.\Big|_{\eta=p}^{\eta=p-W},\quad \hat l_i^{\,A}=\partial f_i^{\,A}/\partial z(\xi,\eta,-z)\|,\quad l_i^{\,B}=\partial f_i^{\,B}/\partial z(\xi,\eta,z)\|,\quad l_i^{\,C}=\partial f_i^{\,C}/\partial z(\xi,\eta,z)\|$$

(1) Strike slip

$$f_i^{\,A}=\begin{pmatrix} u_1=\dfrac{\Theta}{2} & +\dfrac{\alpha}{2}\xi qY_{11} \\ u_2= & \dfrac{\alpha}{2}\dfrac{q}{R} \\ u_3=\dfrac{1-\alpha}{2}\ln(R+\eta) & -\dfrac{\alpha}{2}q^2Y_{11} \end{pmatrix}\qquad f_i^{\,B}=\begin{pmatrix} u_1=-\xi qY_{11}-\Theta-\dfrac{1-\alpha}{\alpha}I_1\sin\delta \\ u_2=-\dfrac{q}{R}\quad+\dfrac{1-\alpha}{\alpha}\dfrac{\tilde y}{R+\tilde d}\sin\delta \\ u_3=q^2Y_{11}\quad-\dfrac{1-\alpha}{\alpha}I_2\sin\delta \end{pmatrix}\qquad \begin{aligned}\Theta&=\tan^{-1}\frac{\xi\eta}{qR}\\ Y_{11}&=\frac{1}{R(R+\eta)}\end{aligned}$$

$$f_i^{\,C}=\begin{pmatrix} u_1=(1-\alpha)\xi Y_{11}\cos\delta & -\alpha\xi qZ_{32} \\ u_2=(1-\alpha)\left(\dfrac{\cos\delta}{R}+2qY_{11}\sin\delta\right) & -\alpha\dfrac{\tilde cq}{R^3} \\ u_3=(1-\alpha)qY_{11}\cos\delta & -\alpha\left(\dfrac{\tilde c\eta}{R^3}-zY_{11}+\xi^2Z_{32}\right) \end{pmatrix}\qquad \begin{aligned} Y_{32}&=\frac{2R+\eta}{R^3(R+\eta)^2}\\ Z_{32}&=\frac{\sin\delta}{R^3}-hY_{32}\\ h&=q\cos\delta-z\,,\quad \tilde c=\tilde d+z=\eta\sin\delta-h\end{aligned}$$

where, $q=y\sin\delta-d\cos\delta,\quad \tilde y=\eta\cos\delta+q\sin\delta,\quad \tilde d=\eta\sin\delta-q\cos\delta,\quad R^2=\xi^2+\eta^2+q^2=X^2+\eta^2\,,$

By differentiation with y-coordinate (refer Appendix “Table of Differentiation of Integrals”)

$$\frac{\partial f_i^A}{\partial z}=\begin{pmatrix}\frac{1}{2}(\tilde{y}X_{11}+\xi Y_{11}\cos\delta) & +\frac{\alpha}{2}\xi\left[\frac{\tilde{y}}{R^3}-(Y_{11}-\xi^2 Y_{32})\cos\delta\right]\\ & \frac{\alpha}{2}\left(\frac{\cos\delta}{R}+\frac{\tilde{d}q}{R^3}\right)\\ -\frac{1-\alpha}{2}\left(\frac{\sin\delta}{R}-qY_{11}\cos\delta\right) & -\frac{\alpha}{2}q\left(\frac{\tilde{y}}{R^3}+\xi^2 Y_{32}\cos\delta\right)\end{pmatrix}=\begin{pmatrix}\frac{1-\alpha}{2}\xi Y_{11}\cos\delta+\frac{\tilde{y}}{2}X_{11} & +\frac{\alpha}{2}\xi F'\\ & \frac{\alpha}{2}E'\\ -\frac{1-\alpha}{2}\left(\frac{\sin\delta}{R}-qY_{11}\cos\delta\right) & -\frac{\alpha}{2}qF'\end{pmatrix}$$

$$E'=\frac{\cos\delta}{R}+\frac{\tilde{d}q}{R^3}\qquad F'=\frac{\tilde{y}}{R^3}+\xi^2 Y_{32}\cos\delta$$

$$\frac{\partial f_i^B}{\partial z}=\begin{pmatrix}-\xi\left[\frac{\tilde{y}}{R^3}-(Y_{11}-\xi^2 Y_{32})\cos\delta\right]-(\tilde{y}X_{11}+\xi Y_{11}\cos\delta) & +\frac{1-\alpha}{\alpha}J_1^z\sin\delta\\ -\left(\frac{\cos\delta}{R}+\frac{\tilde{d}q}{R^3}\right) & +\frac{1-\alpha}{\alpha}J_2^z\sin\delta\\ q\left(\frac{\tilde{y}}{R^3}+\xi^2 Y_{32}\cos\delta\right) & +\frac{1-\alpha}{\alpha}J_3^z\sin\delta\end{pmatrix}=\begin{pmatrix}-\xi F'-\tilde{y}X_{11} & +\frac{1-\alpha}{\alpha}J_1^z\sin\delta\\ -E' & +\frac{1-\alpha}{\alpha}J_2^z\sin\delta\\ qF' & +\frac{1-\alpha}{\alpha}J_3^z\sin\delta\end{pmatrix}$$

$$J_1^z=-\frac{\partial I_1}{\partial z}\qquad J_2^z=\frac{\partial}{\partial z}\left(\frac{\tilde{y}}{R+\tilde{d}}\right)\qquad J_3^z=-\frac{\partial I_2}{\partial z}$$

$$\frac{\partial f_i^C}{\partial z}=\begin{pmatrix}(1-\alpha)\xi\left(\frac{\sin\delta}{R^3}-qY_{32}\cos\delta\right)\cos\delta & -\alpha\xi\left\{\frac{3\tilde{c}\tilde{y}}{R^5}+qY_{32}-(zY_{32}+Z_{32}+Z_0)\cos\delta\right\}\\ (1-\alpha)\left\{\frac{\tilde{d}}{R^3}\cos\delta+2\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\sin\delta\right\} & -\alpha\tilde{c}\left(\frac{\cos\delta}{R^3}+\frac{3\tilde{d}q}{R^5}\right)\\ \multicolumn{2}{l}{(1-\alpha)\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\cos\delta+\alpha\left\{\tilde{c}\left(\frac{\sin\delta}{R^3}-\frac{3\tilde{d}\eta}{R^5}\right)+Y_{11}+z\left(\frac{\sin\delta}{R^3}-qY_{32}\cos\delta\right)-\xi^2\left(\frac{3\tilde{c}}{R^5}\sin\delta+Y_{32}\sin^2\delta-qZ_{53}\cos\delta\right)\right\}}\end{pmatrix}$$

Here, $\frac{\partial f_2^C}{\partial z}=2(1-\alpha)\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\sin\delta+\frac{\tilde{d}}{R^3}\cos\delta-\alpha\left(\frac{\tilde{c}+\tilde{d}}{R^3}\cos\delta+\frac{3\tilde{c}\tilde{d}q}{R^5}\right)$

And since $z=\tilde{c}-\tilde{d}$ and $(Y_{11}-\xi^2 Y_{32})+(Y_{11}-q^2 Y_{32})=\frac{\eta}{R^3}$ (refer (∗ 3) of Appendix)

$$\begin{aligned}\frac{\partial f_3^C}{\partial z}&=\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\cos\delta+\alpha\left\{-\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\cos\delta+\frac{\tilde{c}+z}{R^3}\sin\delta-\frac{3\tilde{c}(\tilde{d}\eta+\xi^2\sin\delta)}{R^5}+Y_{11}-(q\cos\delta-h)qY_{32}\cos\delta-\xi^2(Y_{32}\sin^2\delta-qZ_{53}\cos\delta)\right\}\\ &=\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\cos\delta+\alpha\left\{-\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\cos\delta+\frac{\tilde{c}+z}{R^3}\sin\delta+\frac{3\tilde{c}(\tilde{y}q-R^2\sin\delta)}{R^5}+Y_{11}-(\xi^2\sin^2\delta+q^2\cos^2\delta)Y_{32}+qhY_{32}\cos\delta+\xi^2 qZ_{53}\cos\delta\right\}\\ &=\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\cos\delta+\alpha\left\{-\frac{\eta}{R^3}\cos^2\delta-\frac{q}{R^3}\sin\delta\cos\delta+\frac{z-2\tilde{c}}{R^3}\sin\delta+\frac{3\tilde{c}\tilde{y}q}{R^5}+Y_{11}-\xi^2 Y_{32}\sin^2\delta+(Y_0-q^2 Y_{32})\cos^2\delta+q\cos\delta\,(hY_{32}+\xi^2 Z_{53})\right\}\\ &=\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\cos\delta+\alpha\left\{-\frac{\eta}{R^3}\cos^2\delta-\frac{\tilde{c}+\tilde{d}}{R^3}\sin\delta+\frac{3\tilde{c}\tilde{y}q}{R^5}+Y_{11}-\xi^2 Y_{32}\sin^2\delta+\left(\frac{\eta}{R^3}-Y_{11}\right)\cos^2\delta-q\cos\delta\left(\frac{\sin\delta}{R^3}-hY_{32}-\xi^2 Z_{53}\right)\right\}\\ &=\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\cos\delta+\alpha\left\{-\frac{\tilde{c}+\tilde{d}}{R^3}\sin\delta+\frac{3\tilde{c}\tilde{y}q}{R^5}+Y_{11}\sin^2\delta-\xi^2 Y_{32}\sin^2\delta-qZ_0\cos\delta\right\}\\ &=\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\cos\delta-\alpha\left\{\frac{\tilde{c}+\tilde{d}}{R^3}\sin\delta-\frac{3\tilde{c}\tilde{y}q}{R^5}-Y_0\sin^2\delta+qZ_0\cos\delta\right\}\end{aligned}$$

The above three vectors correspond to the contents of the row of Strike-Slip in Table 9.
(Evaluation of J_1^z et al. will be done in the later section)

(2) Dip slip

$$f_i^A=\begin{pmatrix}u_1= & \frac{\alpha}{2}\frac{q}{R}\\ u_2=\frac{\Theta}{2} & +\frac{\alpha}{2}\eta qX_{11}\\ u_3=\frac{1-\alpha}{2}\ln(R+\xi) & -\frac{\alpha}{2}q^2 X_{11}\end{pmatrix}\qquad f_i^B=\begin{pmatrix}u_1=-\frac{q}{R} & +\frac{1-\alpha}{\alpha}I_3\sin\delta\cos\delta\\ u_2=-\eta qX_{11}-\Theta & -\frac{1-\alpha}{\alpha}\frac{\xi}{R+\tilde{d}}\sin\delta\cos\delta\\ u_3=q^2 X_{11} & +\frac{1-\alpha}{\alpha}I_4\sin\delta\cos\delta\end{pmatrix}$$

$$\Theta=\tan^{-1}\frac{\xi\eta}{qR}\qquad X_{11}=\frac{1}{R(R+\xi)}$$

$$f_i^C=\begin{pmatrix}u_1= & (1-\alpha)\frac{\cos\delta}{R}-qY_{11}\sin\delta & -\alpha\frac{\tilde{c}q}{R^3}\\ u_2= & (1-\alpha)\tilde{y}X_{11} & -\alpha\tilde{c}\eta qX_{32}\\ u_3= & -\tilde{d}X_{11}-\xi Y_{11}\sin\delta & -\alpha\tilde{c}(X_{11}-q^2 X_{32})\end{pmatrix}\qquad Y_{11}=\frac{1}{R(R+\eta)}\qquad X_{32}=\frac{2R+\xi}{R^3(R+\xi)^2}\qquad \tilde{c}=\tilde{d}+z$$

where, $q=y\sin\delta-d\cos\delta$, $\tilde{y}=\eta\cos\delta+q\sin\delta$, $\tilde{d}=\eta\sin\delta-q\cos\delta$, $R^2=\xi^2+\eta^2+q^2=X^2+\eta^2$,

By differentiation with y-coordinate (refer Appendix “Table of Differentiation of Integrals”)

$$\frac{\partial f_i^A}{\partial z}=\begin{pmatrix} & \frac{\alpha}{2}\left(\frac{\cos\delta}{R}+\frac{\tilde{d}q}{R^3}\right)\\ \frac{1}{2}(\tilde{y}X_{11}+\xi Y_{11}\cos\delta) & +\frac{\alpha}{2}\left[(2\eta\cos\delta-\tilde{y})X_{11}+\tilde{d}\eta qX_{32}\right]\\ -\frac{1-\alpha}{2}\tilde{d}X_{11} & -\frac{\alpha}{2}q(2X_{11}\cos\delta+\tilde{d}qX_{32})\end{pmatrix}=\begin{pmatrix} & \frac{\alpha}{2}E'\\ \frac{1-\alpha}{2}\tilde{y}X_{11}+\frac{\xi}{2}Y_{11}\cos\delta & +\frac{\alpha}{2}\eta G'\\ -\frac{1-\alpha}{2}\tilde{d}X_{11} & -\frac{\alpha}{2}qG'\end{pmatrix}$$

$$E'=\frac{\cos\delta}{R}+\frac{\tilde{d}q}{R^3}\qquad G'=2X_{11}\cos\delta+\tilde{d}qX_{32}$$

$$\frac{\partial f_i^B}{\partial z} = \begin{pmatrix} -\left(\frac{\cos\delta}{R} + \frac{\tilde{d}q}{R^3}\right) & -\frac{1-\alpha}{\alpha} J_4^z \sin\delta\cos\delta \\ -[(2\eta\cos\delta - \tilde{y})X_{11} + \tilde{d}\eta q X_{32}] - (\tilde{y}X_{11} + \xi Y_{11}\cos\delta) & -\frac{1-\alpha}{\alpha} J_5^z \sin\delta\cos\delta \\ q(2X_{11}\cos\delta + \tilde{d}qX_{32}) & -\frac{1-\alpha}{\alpha} J_6^z \sin\delta\cos\delta \end{pmatrix} = \begin{pmatrix} -E' & -\frac{1-\alpha}{\alpha} J_4^z \sin\delta\cos\delta \\ -\eta G' - \xi Y_{11}\cos\delta & -\frac{1-\alpha}{\alpha} J_5^z \sin\delta\cos\delta \\ qG' & -\frac{1-\alpha}{\alpha} J_6^z \sin\delta\cos\delta \end{pmatrix} \quad \begin{aligned} J_4^z &= -\frac{\partial I_3}{\partial z} \\ J_5^z &= \frac{\partial}{\partial z}\left(\frac{\xi}{R+\tilde{d}}\right) \\ J_6^z &= -\frac{\partial I_4}{\partial z} \end{aligned}$$

$$\frac{\partial f_i^C}{\partial z} = \begin{pmatrix} (1-\alpha)\frac{\tilde{d}}{R^3}\cos\delta - \left(\frac{\tilde{y}}{R^3} - Y_0\cos\delta\right)\sin\delta & -\alpha\tilde{c}\left(\frac{\cos\delta}{R^3} + \frac{3\tilde{d}q}{R^5}\right) \\ (1-\alpha)\tilde{y}\tilde{d}X_{32} & -\alpha\tilde{c}[(\tilde{y} - 2q\sin\delta)X_{32} + \tilde{d}\eta q X_{53}] \\ X_{11} - \tilde{d}^2 X_{32} - \xi\left(\frac{\sin\delta}{R^3} - qY_{32}\cos\delta\right)\sin\delta & -\alpha\tilde{c}(\tilde{d}X_{32} - 2qX_{32}\cos\delta - \tilde{d}q^2 X_{53}) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\tilde{d}\cos\delta - \tilde{y}\sin\delta}{R^3} + Y_0\sin\delta\cos\delta - \alpha\left(\frac{\tilde{c}+\tilde{d}}{R^3}\cos\delta + \frac{3\tilde{c}\tilde{d}q}{R^5}\right) \\ (1-\alpha)\tilde{y}\tilde{d}X_{32} \quad -\alpha\tilde{c}[(\tilde{y} - 2q\sin\delta)X_{32} + \tilde{d}\eta q X_{53}] \\ -\xi P'\sin\delta + X_{11} - \tilde{d}^2 X_{32} \quad -\alpha\tilde{c}[(\tilde{d} - 2q\cos\delta)X_{32} - \tilde{d}q^2 X_{53}] \end{pmatrix} \quad \begin{aligned} &\tilde{d}\cos\delta - \tilde{y}\sin\delta = -q \\ &P' = \frac{\sin\delta}{R^3} - qY_{32}\cos\delta \end{aligned}$$

The above three vectors correspond to the contents of the row of Dip-Slip in Table 9.
(Evaluation of J_4^z et al. will be done in the later section)

(3) Tensile

$$f_i^A = \begin{pmatrix} u_1 = -\frac{1-\alpha}{2}\ln(R+\eta) - \frac{\alpha}{2}q^2 Y_{11} \\ u_2 = -\frac{1-\alpha}{2}\ln(R+\xi) - \frac{\alpha}{2}q^2 X_{11} \\ u_3 = \frac{\Theta}{2} \quad -\frac{\alpha}{2}q(\eta X_{11} + \xi Y_{11}) \end{pmatrix} \quad f_i^B = \begin{pmatrix} u_1 = q^2 Y_{11} & -\frac{1-\alpha}{\alpha} I_3 \sin^2\delta \\ u_2 = q^2 X_{11} & +\frac{1-\alpha}{\alpha}\frac{\xi}{R+\tilde{d}}\sin^2\delta \\ u_3 = q(\eta X_{11} + \xi Y_{11}) - \Theta & -\frac{1-\alpha}{\alpha} I_4 \sin^2\delta \end{pmatrix} \quad \begin{aligned} \Theta &= \tan^{-1}\frac{\xi\eta}{qR} \\ X_{11} &= \frac{1}{R(R+\xi)} \\ Y_{11} &= \frac{1}{R(R+\eta)} \end{aligned}$$

$$f_i^C = \begin{pmatrix} u_1 = -(1-\alpha)\left(\frac{\sin\delta}{R} + qY_{11}\cos\delta\right) - \alpha(zY_{11} - q^2 Z_{32}) \\ u_2 = (1-\alpha)2\xi Y_{11}\sin\delta + \tilde{d}X_{11} \quad -\alpha\tilde{c}(X_{11} - q^2 X_{32}) \\ u_3 = (1-\alpha)(\tilde{y}X_{11} + \xi Y_{11}\cos\delta) \ +\alpha q(\tilde{c}\eta X_{32} + \xi Z_{32}) \end{pmatrix} \quad \begin{aligned} &X_{32} = \frac{2R+\xi}{R^3(R+\xi)^2}\ , \quad Y_{32} = \frac{2R+\eta}{R^3(R+\eta)^2} \\ &Z_{32} = \frac{\sin\delta}{R^3} - hY_{32} \\ &h = q\cos\delta - z\ , \quad \tilde{c} = \tilde{d} + z = \eta\sin\delta - h \end{aligned}$$

where, $q = y\sin\delta - d\cos\delta$, $\tilde{y} = \eta\cos\delta + q\sin\delta$, $\tilde{d} = \eta\sin\delta - q\cos\delta$, $R^2 = \xi^2 + \eta^2 + q^2 = X^2 + \eta^2$,

By differentiation with y-coordinate (refer Appendix "Table of Differentiation of Integrals")

$$\frac{\partial f_i^A}{\partial z} = \begin{pmatrix} \frac{1-\alpha}{2}\left(\frac{\sin\delta}{R} - qY_{11}\cos\delta\right) & -\frac{\alpha}{2}q\left(\frac{\tilde{y}}{R^3} + \xi^2 Y_{32}\cos\delta\right) \\ \frac{1-\alpha}{2}\tilde{d}X_{11} & -\frac{\alpha}{2}q(2X_{11}\cos\delta + \tilde{d}qX_{32}) \\ \frac{1}{2}(\tilde{y}X_{11} + \xi Y_{11}\cos\delta) & -\frac{\alpha}{2}\left[(\tilde{y} - 2q\sin\delta)X_{11} + \tilde{d}\eta q X_{32} + \xi\left(\frac{\tilde{y}}{R^3} - (Y_{11} - \xi^2 Y_{32})\cos\delta\right)\right] \end{pmatrix}$$

Since $(Y_{11} - \xi^2 Y_{32}) + (Y_{11} - q^2 Y_{32}) = \frac{\eta}{R^3}$ (refer (∗ 3) of Appendix)

$$\begin{aligned} \frac{\partial f_3^A}{\partial z} &= \frac{1}{2}(\tilde{y}X_{11} + \xi Y_{11}\cos\delta) - \frac{\alpha}{2}\left\{\tilde{y}X_{11} - 2qX_{11}\sin\delta + \tilde{d}\eta q X_{32} + \xi\left(\frac{\tilde{y}}{R^3} - \left[\frac{\eta}{R^3} - (Y_{11} - q^2 Y_{32})\right]\cos\delta\right)\right\} \\ &= \frac{1-\alpha}{2}(\tilde{y}X_{11} + \xi Y_{11}\cos\delta) + \frac{\alpha}{2}\left\{2qX_{11}\sin\delta - \tilde{d}\eta q X_{32} - \frac{\xi q}{R^3}\sin\delta + \xi q^2 Y_{32}\cos\delta\right\} = \frac{1-\alpha}{2}(\tilde{y}X_{11} + \xi Y_{11}\cos\delta) + \frac{\alpha}{2}qH' \\ &= \frac{1-\alpha}{2}(\tilde{y}X_{11} + \xi Y_{11}\cos\delta) + \frac{\alpha}{2}qH' \end{aligned}$$

Here,
$$\begin{aligned} H' &= 2X_{11}\sin\delta - \tilde{d}\eta X_{32} - \frac{\xi}{R^3}\sin\delta + \xi q Y_{32}\cos\delta \\ &= 2X_{11}\sin\delta - (\eta\sin\delta - q\cos\delta)\eta X_{32} - \frac{\xi}{R^3}\sin\delta + \xi q Y_{32}\cos\delta \\ &= -\left(\eta^2 X_{32} + \frac{\xi}{R^3} - 2X_{11}\right)\sin\delta + \eta q X_{32}\cos\delta + \xi q Y_{32}\cos\delta \\ &= \frac{2R+\xi}{R^3(R+\xi)^2}(R^2 - \xi^2 - \eta^2)\sin\delta + \eta q X_{32}\cos\delta + \xi q Y_{32}\cos\delta \\ &= q^2 X_{32}\sin\delta + \eta q X_{32}\cos\delta + \xi q Y_{32}\cos\delta \\ &= (\eta\cos\delta + q\sin\delta)qX_{32} + \xi q Y_{32}\cos\delta \\ &= \tilde{y}qX_{32} + \xi q Y_{32}\cos\delta \end{aligned}$$

Therefore,

$$\frac{\partial f_i^A}{\partial z}=\begin{pmatrix}\frac{1-\alpha}{2}\left(\frac{\sin\delta}{R}-qY_{11}\cos\delta\right)-\frac{\alpha}{2}qF' \\ \frac{1-\alpha}{2}\tilde{d}X_{11} \qquad -\frac{\alpha}{2}qG' \\ \frac{1-\alpha}{2}(\tilde{y}X_{11}+\xi Y_{11}\cos\delta)+\frac{\alpha}{2}qH'\end{pmatrix}\qquad \begin{aligned}F'&=\frac{\tilde{y}}{R^3}+\xi^2Y_{32}\cos\delta\\ G'&=2X_{11}\cos\delta+\tilde{d}qX_{32}\\ H'&=\tilde{y}qX_{32}+\xi qY_{32}\cos\delta\end{aligned}$$

$$\frac{\partial f_i^B}{\partial z}=\begin{pmatrix} q\left(\frac{\tilde{y}}{R^3}+\xi^2Y_{32}\cos\delta\right) +\frac{1-\alpha}{\alpha}J_4^z\sin^2\delta \\ q(2X_{11}\cos\delta+\tilde{d}qX_{32}) +\frac{1-\alpha}{\alpha}J_5^z\sin^2\delta \\ (\tilde{y}-2q\sin\delta)X_{11}+\tilde{d}\eta qX_{32}+\xi\left(\frac{\tilde{y}}{R^3}-(Y_{11}-\xi^2Y_{32})\cos\delta\right)-(\tilde{y}X_{11}+\xi Y_{11}\cos\delta)+\frac{1-\alpha}{\alpha}J_6^z\sin^2\delta\end{pmatrix}\qquad \begin{aligned}J_4^z&=-\frac{\partial I_3}{\partial z}\\ J_5^z&=\frac{\partial}{\partial z}\left(\frac{\xi}{R+\tilde{d}}\right)\\ J_6^z&=-\frac{\partial I_4}{\partial z}\end{aligned}$$

$$=\begin{pmatrix} q\left(\frac{\tilde{y}}{R^3}+\xi^2Y_{32}\cos\delta\right) +\frac{1-\alpha}{\alpha}J_4^z\sin^2\delta \\ q(2X_{11}\cos\delta+\tilde{d}qX_{32}) +\frac{1-\alpha}{\alpha}J_5^z\sin^2\delta \\ -2qX_{11}\sin\delta+\tilde{d}\eta qX_{32}+\frac{\xi q}{R^3}\sin\delta-\xi q^2Y_{32}\cos\delta+\frac{1-\alpha}{\alpha}J_6^z\sin^2\delta\end{pmatrix}=\begin{pmatrix} qF'+\frac{1-\alpha}{\alpha}J_4^z\sin^2\delta \\ qG'+\frac{1-\alpha}{\alpha}J_5^z\sin^2\delta \\ -qH'+\frac{1-\alpha}{\alpha}J_6^z\sin^2\delta\end{pmatrix}$$

$$\frac{\partial f_i^C}{\partial z}=\begin{pmatrix} -(1-\alpha)\left\{\frac{\tilde{d}}{R^3}\sin\delta+\left(\frac{\tilde{y}}{R^3}-Y_0\cos\delta\right)\cos\delta\right\} -\alpha\left\{Y_{11}+z\left(\frac{\sin\delta}{R^3}-qY_{32}\cos\delta\right)-q\left[\frac{3\tilde{c}\tilde{y}}{R^5}+qY_{32}-(zY_{32}+Z_0)\cos\delta\right]\right\} \\ 2(1-\alpha)\xi\left(\frac{\sin\delta}{R^3}-qY_{32}\cos\delta\right)\sin\delta-X_{11}+\tilde{d}^2X_{32}-\alpha\tilde{c}(\tilde{d}X_{32}-2qX_{32}\cos\delta-\tilde{d}q^2X_{53}) \\ (1-\alpha)\left\{\tilde{y}\tilde{d}X_{32}+\xi\left(\frac{\sin\delta}{R^3}-qY_{32}\cos\delta\right)\cos\delta\right\}+\alpha\left\{\tilde{c}[(\tilde{y}-2q\sin\delta)X_{32}+\tilde{d}\eta qX_{53}]+\xi\left[\frac{3\tilde{c}\tilde{y}}{R^5}+qY_{32}-(zY_{32}+Z_{32}+Z_0)\cos\delta\right]\right\}\end{pmatrix}$$

$$=\begin{pmatrix} -(1-\alpha)\left(\frac{\eta}{R^3}-Y_0\cos^2\delta\right) -\alpha\left(\frac{z}{R^3}\sin\delta-\frac{3\tilde{c}\tilde{y}q}{R^5}+Y_{11}-q^2Y_{32}+qZ_0\cos\delta\right) \\ 2(1-\alpha)\xi P'-X_{11}+\tilde{d}^2X_{32}-\alpha\tilde{c}[(\tilde{d}-2q\cos\delta)X_{32}-\tilde{d}q^2X_{53}] \\ (1-\alpha)[\xi P'\cos\delta+\tilde{y}\tilde{d}X_{32}]+\alpha\tilde{c}[(\tilde{y}-2q\sin\delta)X_{32}+\tilde{d}\eta qX_{53}]+\alpha\xi Q'\end{pmatrix}\qquad \begin{aligned}P'&=\frac{\sin\delta}{R^3}-qY_{32}\cos\delta\\ Q'&=\frac{3\tilde{c}\tilde{y}}{R^5}+qY_{32}-(zY_{32}+Z_{32}+Z_0)\cos\delta\end{aligned}$$

Since $(Y_{11}-\xi^2Y_{32})+(Y_{11}-q^2Y_{32})=\frac{\eta}{R^3}$ (refer (∗ 3) of Appendix)

$$\begin{aligned}\frac{\partial f_1^C}{\partial z}&=-\frac{\eta}{R^3}+Y_0\cos^2\delta-\alpha\left(\frac{z}{R^3}\sin\delta-\frac{3\tilde{c}\tilde{y}q}{R^5}-\frac{\eta}{R^3}+Y_0\cos^2\delta+Y_{11}-q^2Y_{32}+qZ_0\cos\delta\right)\\ &=-\frac{\eta}{R^3}+Y_0\cos^2\delta-\alpha\left(\frac{z}{R^3}\sin\delta-\frac{3\tilde{c}\tilde{y}q}{R^5}-Y_0\sin^2\delta+qZ_0\cos\delta\right)\end{aligned}$$

The above three vectors correspond to the contents of the row of Tensile in Table 9.
(Evaluation of J_4^z et al. will be done in the next section)

[IV] Evaluation of $J_1^x-J_6^x$, $J_1^y-J_6^y$ and $J_1^z-J_6^z$

$J_1^x-J_6^x$, $J_1^y-J_6^y$ and $J_1^z-J_6^z$ can be exaluated as follows (refer Appendix "Table of Differentiation of Integrals")

$$I_1=-\frac{\xi}{R+\tilde{d}}\cos\delta-I_4\sin\delta,\qquad I_2=\ln(R+\tilde{d})+I_3\sin\delta$$

$$I_3=\frac{1}{\cos\delta}\frac{\tilde{y}}{R+\tilde{d}}-\frac{1}{\cos^2\delta}[\ln(R+\eta)-\sin\delta\ln(R+\tilde{d})]\qquad \left(I_3=\frac{1}{2}\left[\frac{\eta}{R+\tilde{d}}+\frac{\tilde{y}q}{(R+\tilde{d})^2}-\ln(R+\eta)\right]\quad \text{if }\cos\delta=0\right)$$

$$I_4=\frac{\sin\delta}{\cos\delta}\frac{\xi}{R+\tilde{d}}+\frac{2}{\cos^2\delta}\tan^{-1}\frac{\eta(X+q\cos\delta)+X(R+X)\sin\delta}{\xi(R+X)\cos\delta}\qquad \left(I_4=\frac{\xi\tilde{y}}{2(R+\tilde{d})^2}\quad \text{if }\cos\delta=0\right)$$

(a) In case of $\cos\delta\neq 0$

For x-derivative

$$J_2^x=\frac{\partial}{\partial x}\left(-\frac{\tilde{y}}{R+\tilde{d}}\right)=\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11}\equiv \boldsymbol{J_2}$$

$$J_5^x=\frac{\partial}{\partial x}\left(-\frac{\xi}{R+\tilde{d}}\right)=-\left(\tilde{d}+\frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11}\equiv \boldsymbol{J_5}$$

$$J_4^x=\frac{\partial I_3}{\partial x}=\frac{1}{\cos\delta}\frac{\partial}{\partial x}\left(\frac{\tilde{y}}{R+\tilde{d}}\right)-\frac{1}{\cos^2\delta}\frac{\partial}{\partial x}[\ln(R+\eta)-\sin\delta\ln(R+\tilde{d})]=-\frac{1}{\cos\delta}\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11}-\frac{\xi}{\cos^2\delta}(Y_{11}-D_{11}\sin\delta)\equiv \boldsymbol{J_4}$$

$$J_6^x=\frac{\partial I_4}{\partial x}=\frac{\sin\delta}{\cos\delta}\frac{\partial}{\partial x}\left(\frac{\xi}{R+\tilde{d}}\right)+\frac{2}{\cos^2\delta}\frac{\partial}{\partial x}\tan^{-1}\frac{\eta(X+q\cos\delta)+X(R+X)\sin\delta}{\xi(R+X)\cos\delta}=\frac{\sin\delta}{\cos\delta}\left(\tilde{d}+\frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11}+\frac{1}{\cos^2\delta}(qY_{11}-\tilde{y}D_{11})\equiv \boldsymbol{J_6}$$

$$J_3^x=\frac{\partial I_2}{\partial x}=\frac{\partial}{\partial x}\ln(R+\tilde{d})+\frac{\partial I_3}{\partial x}\sin\delta=\xi D_{11}-\frac{\sin\delta}{\cos\delta}\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11}-\frac{\xi\sin\delta}{\cos^2\delta}(Y_{11}-D_{11}\sin\delta)=-\frac{\sin\delta}{\cos\delta}\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11}+\frac{\xi}{\cos^2\delta}(D_{11}-Y_{11}\sin\delta)\equiv \boldsymbol{J_3}$$

$$J_1^x=\frac{\partial I_1}{\partial x}=\frac{\partial}{\partial x}\left(-\frac{\xi}{R+\tilde{d}}\right)\cos\delta-\frac{\partial I_4}{\partial x}\sin\delta=-\frac{1}{\cos\delta}\left(\tilde{d}+\frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11}-\frac{\sin\delta}{\cos^2\delta}(qY_{11}-\tilde{y}D_{11})\equiv \boldsymbol{J_1}$$

For y-derivative

$$J_2^y = \frac{\partial}{\partial y}\left(\frac{\tilde{y}}{R+\tilde{d}}\right) = \frac{1}{R+\tilde{d}} - \frac{\tilde{y}^2}{R+\tilde{d}}D_{11} = \frac{1}{R+\tilde{d}} + \left(\tilde{d}D_{11} + \boldsymbol{J_5}\right) = \frac{1}{R} + \boldsymbol{J_5}$$

$$J_5^y = \frac{\partial}{\partial y}\left(-\frac{\xi}{R+\tilde{d}}\right) = \frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} = \boldsymbol{J_2}$$

$$J_4^y = \frac{\partial I_3}{\partial y} = \frac{1}{\cos\delta}\frac{\partial}{\partial y}\left(\frac{\tilde{y}}{R+\tilde{d}}\right) - \frac{1}{\cos^2\delta}\frac{\partial}{\partial y}\left[\ln(R+\eta) - \sin\delta\ln(R+\tilde{d})\right] = \frac{1}{\cos\delta}\left(\frac{1}{R+\tilde{d}} - \frac{\tilde{y}^2}{R+\tilde{d}}D_{11}\right) - \frac{1}{\cos^2\delta}\left(\frac{\cos\delta}{R} + qY_{11}\sin\delta - \tilde{y}D_{11}\sin\delta\right)$$

$$= -\frac{1}{\cos\delta}\left(\tilde{d} + \frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11} - \frac{\sin\delta}{\cos^2\delta}\left(qY_{11} - \tilde{y}D_{11}\right) = \boldsymbol{J_1}$$

$$J_6^y = \frac{\partial I_4}{\partial y} = \frac{\sin\delta}{\cos\delta}\frac{\partial}{\partial y}\left(\frac{\xi}{R+\tilde{d}}\right) + \frac{2}{\cos^2\delta}\frac{\partial}{\partial y}\tan^{-1}\frac{\eta(X+q\cos\delta)+X(R+X)\sin\delta}{\xi(R+X)\cos\delta} = -\frac{\sin\delta}{\cos\delta}\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} + \frac{\xi}{\cos^2\delta}\left(D_{11} - Y_{11}\sin\delta\right) = \boldsymbol{J_3}$$

$$J_3^y = \frac{\partial I_2}{\partial y} = \frac{\partial}{\partial y}\ln(R+\tilde{d}) + \frac{\partial I_3}{\partial y}\sin\delta = \tilde{y}D_{11} - \frac{\sin\delta}{\cos\delta}\left(\tilde{d} + \frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11} - \frac{\sin^2\delta}{\cos^2\delta}\left(qY_{11} - \tilde{y}D_{11}\right)$$

$$= -\frac{\sin\delta}{\cos\delta}\left(\tilde{d} + \frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11} - \frac{\sin^2\delta}{\cos^2\delta}qY_{11} + \frac{1}{\cos^2\delta}\tilde{y}D_{11} = qY_{11} - \frac{\sin\delta}{\cos\delta}\left(\tilde{d} + \frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11} - \frac{1}{\cos^2\delta}\left(qY_{11} - \tilde{y}D_{11}\right) = qY_{11} - \boldsymbol{J_6}$$

$$J_1^y = -\frac{\partial I_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\xi}{R+\tilde{d}}\right)\cos\delta + \frac{\partial I_4}{\partial y}\sin\delta = -\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11}\cos\delta - \frac{\sin^2\delta}{\cos\delta}\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} + \frac{\xi\sin\delta}{\cos^2\delta}\left(D_{11} - Y_{11}\sin\delta\right)$$

$$= -\frac{1}{\cos\delta}\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} - \frac{\sin^2\delta}{\cos^2\delta}\xi Y_{11} + \frac{\sin\delta}{\cos^2\delta}\xi D_{11} = \xi Y_{11} - \frac{1}{\cos\delta}\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} - \frac{\xi}{\cos^2\delta}\left(Y_{11} - D_{11}\sin\delta\right) = \xi Y_{11} + \boldsymbol{J_4}$$

For z-derivative

$$J_2^z = \frac{\partial}{\partial z}\left(\frac{\tilde{y}}{R+\tilde{d}}\right) = \tilde{y}D_{11}$$

$$J_5^z = \frac{\partial}{\partial z}\left(\frac{\xi}{R+\tilde{d}}\right) = \xi D_{11}$$

$$J_4^z = -\frac{\partial I_3}{\partial z} = -\frac{1}{\cos\delta}\frac{\partial}{\partial z}\left(\frac{\tilde{y}}{R+\tilde{d}}\right) + \frac{1}{\cos^2\delta}\frac{\partial}{\partial z}\left[\ln(R+\eta) - \sin\delta\ln(R+\tilde{d})\right] = -\frac{1}{\cos\delta}\left(\tilde{y}D_{11} - qY_{11}\right) \equiv \boldsymbol{K_3}$$

$$J_6^z = -\frac{\partial I_4}{\partial z} = -\frac{\sin\delta}{\cos\delta}\frac{\partial}{\partial z}\left(\frac{\xi}{R+\tilde{d}}\right) - \frac{2}{\cos^2\delta}\frac{\partial}{\partial z}\tan^{-1}\frac{\eta(X+q\cos\delta)+X(R+X)\sin\delta}{\xi(R+X)\cos\delta} = \frac{\xi}{\cos\delta}\left(Y_{11} - D_{11}\sin\delta\right) \equiv \boldsymbol{K_4}$$

$$J_3^z = -\frac{\partial I_2}{\partial z} = -\frac{\partial}{\partial z}\ln(R+\tilde{d}) - \frac{\partial I_3}{\partial z}\sin\delta = \frac{1}{R} - \frac{\sin\delta}{\cos\delta}\left(\tilde{y}D_{11} - qY_{11}\right) \equiv \boldsymbol{K_2}$$

$$J_1^z = -\frac{\partial I_1}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\xi}{R+\tilde{d}}\right)\cos\delta + \frac{\partial I_4}{\partial z}\sin\delta = \xi D_{11}\cos\delta - \frac{\xi\sin\delta}{\cos\delta}\left(Y_{11} - D_{11}\sin\delta\right) = \frac{\xi}{\cos\delta}\left(D_{11} - Y_{11}\sin\delta\right) \equiv \boldsymbol{K_1}$$

Namely

$$J_1^x = \boldsymbol{J_1} = -\frac{1}{\cos\delta}\left(\tilde{d} + \frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11} - \frac{\sin\delta}{\cos^2\delta}\left(qY_{11} - \tilde{y}D_{11}\right) \qquad J_1^y = \xi Y_{11} + \boldsymbol{J_4} \qquad J_1^z = \boldsymbol{K_1} = \frac{\xi}{\cos\delta}\left(D_{11} - Y_{11}\sin\delta\right)$$

$$J_2^x = \boldsymbol{J_2} = \frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} \qquad J_2^y = \frac{1}{R} + \boldsymbol{J_5} \qquad J_2^z = \tilde{y}D_{11}$$

$$J_3^x = \boldsymbol{J_3} = -\frac{\sin\delta}{\cos\delta}\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} + \frac{\xi}{\cos^2\delta}\left(D_{11} - Y_{11}\sin\delta\right) \qquad J_3^y = qY_{11} - \boldsymbol{J_6} \qquad J_3^z = \boldsymbol{K_2} = \frac{1}{R} - \frac{\sin\delta}{\cos\delta}\left(\tilde{y}D_{11} - qY_{11}\right)$$

$$J_4^x = \boldsymbol{J_4} = -\frac{1}{\cos\delta}\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} - \frac{\xi}{\cos^2\delta}\left(Y_{11} - D_{11}\sin\delta\right) \qquad J_4^y = \boldsymbol{J_1} \qquad J_4^z = \boldsymbol{K_3} = -\frac{1}{\cos\delta}\left(\tilde{y}D_{11} - qY_{11}\right)$$

$$J_5^x = \boldsymbol{J_5} = -\left(\tilde{d} + \frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11} \qquad J_5^y = \boldsymbol{J_2} \qquad J_5^z = \xi D_{11}$$

$$J_6^x = \boldsymbol{J_6} = \frac{\sin\delta}{\cos\delta}\left(\tilde{d} + \frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11} + \frac{1}{\cos^2\delta}\left(qY_{11} - \tilde{y}D_{11}\right) \qquad J_6^y = \boldsymbol{J_3} \qquad J_6^z = \boldsymbol{K_4} = \frac{\xi}{\cos\delta}\left(Y_{11} - D_{11}\sin\delta\right)$$

And there are following inter-relations

$$\boldsymbol{J_1} = \boldsymbol{J_5}\cos\delta - \boldsymbol{J_6}\sin\delta \qquad \boldsymbol{J_3} = \frac{1}{\cos\delta}\left(\boldsymbol{K_1} - \boldsymbol{J_2}\sin\delta\right) \qquad \boldsymbol{K_2} = \frac{1}{R} + \boldsymbol{K_3}\sin\delta$$

$$\boldsymbol{J_4} = -\xi Y_{11} - \boldsymbol{J_2}\cos\delta + \boldsymbol{J_3}\sin\delta \qquad \boldsymbol{J_6} = \frac{1}{\cos\delta}\left(\boldsymbol{K_3} - \boldsymbol{J_5}\sin\delta\right) \qquad \boldsymbol{K_4} = \xi Y_{11}\cos\delta - \boldsymbol{K_1}\sin\delta$$

(b) In case of $\cos\delta = 0$ $\quad(\sin\delta = \pm 1,\ \tilde{y} = q\sin\delta = \pm q,\ \tilde{d} = \eta\sin\delta = \pm\eta)$

In this case, $Y_{11} = D_{11}\sin\delta = \pm D_{11}$ because $\dfrac{1}{R(R+\eta)} = \dfrac{1}{R(R-\tilde{d})} = -\dfrac{1}{R(R+\tilde{d})} + \dfrac{2}{R^2-\eta^2}$ when $\sin\delta = -1$

For x-derivative

$$J_2^x = \frac{\partial}{\partial x}\left(-\frac{\tilde{y}}{R+\tilde{d}}\right) = \frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} \equiv \boldsymbol{J_2}$$

$$J_5^x = \frac{\partial}{\partial x}\left(-\frac{\xi}{R+\tilde{d}}\right) = -\left(\tilde{d} + \frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11} = -\frac{(R+\tilde{d})\tilde{d} + q^2}{R(R+\tilde{d})^2} = -\frac{1}{R+\tilde{d}}\left(1 - \xi^2 D_{11}\right) \equiv \boldsymbol{J_5}$$

$$J_4^x = \frac{\partial I_3}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\eta}{R+\tilde{d}} + \frac{\tilde{y}q}{\left(R+\tilde{d}\right)^2} - \ln(R+\eta)\right] = -\frac{1}{2}\left[\frac{\xi\eta}{R+\tilde{d}}D_{11} + \frac{2\xi\tilde{y}q}{\left(R+\tilde{d}\right)^2}D_{11} + \xi Y_{11}\right] = -\xi Y_{11} - \frac{\xi\sin\delta}{2}\left[\frac{\tilde{d}}{R+\tilde{d}} + \frac{2q^2}{\left(R+\tilde{d}\right)^2} - 1\right]D_{11}$$

$$= -\xi Y_{11} - \frac{\xi\sin\delta}{2}\left[\frac{\tilde{d}}{R\left(R+\tilde{d}\right)^2} - \frac{1}{R\left(R+\tilde{d}\right)} + \frac{2q^2}{\left(R+\tilde{d}\right)^2}D_{11}\right] = -\xi Y_{11} + \frac{\xi\sin\delta}{\left(R+\tilde{d}\right)^2}\left(\frac{1}{2} - q^2 D_{11}\right) \equiv \boldsymbol{J_4}$$

$$J_6^x = \frac{\partial I_4}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\frac{\xi\tilde{y}}{\left(R+\tilde{d}\right)^2} = \frac{\tilde{y}}{(R+\tilde{d})^2}\left(\frac{1}{2} - \xi^2 D_{11}\right) \equiv \boldsymbol{J_6}$$

$$J_3^x = \frac{\partial I_2}{\partial x} = \frac{\partial}{\partial x}\ln\left(R+\tilde{d}\right) + \frac{\partial I_3}{\partial x}\sin\delta = \xi D_{11} - \xi Y_{11}\sin\delta + \frac{\xi}{(R+\tilde{d})^2}\left(\frac{1}{2} - q^2 D_{11}\right) = \frac{\xi}{\left(R+\tilde{d}\right)^2}\left(\frac{1}{2} - q^2 D_{11}\right) \equiv \boldsymbol{J_3}$$

$$J_1^x = \frac{\partial I_1}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{\xi}{R+\tilde{d}}\right)\cos\delta - \frac{\partial I_4}{\partial x}\sin\delta = -\frac{\tilde{y}}{(R+\tilde{d})^2}\left(\frac{1}{2} - \xi^2 D_{11}\right)\sin\delta = -\frac{q}{(R+\tilde{d})^2}\left(\frac{1}{2} - \xi^2 D_{11}\right) \equiv \boldsymbol{J_1}$$

For y-derivative

$$J_2^y = \frac{\partial}{\partial y}\left(\frac{\tilde{y}}{R+\tilde{d}}\right) = \frac{1}{R+\tilde{d}} - \frac{\tilde{y}^2}{R+\tilde{d}}D_{11} = \frac{1}{R+\tilde{d}} + \left(\tilde{d}D_{11} + \boldsymbol{J_5}\right) = \frac{1}{R+\tilde{d}} + \frac{\tilde{d}}{R\left(R+\tilde{d}\right)} + \boldsymbol{J_5} = \frac{1}{R} + \boldsymbol{J_5}$$

$$J_5^y = \frac{\partial}{\partial y}\left(-\frac{\xi}{R+\tilde{d}}\right) = \frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} = \boldsymbol{J_2}$$

$$J_4^y = \frac{\partial I_3}{\partial y} = \frac{1}{2}\left\{\frac{\cos\delta}{R+\tilde{d}} - \frac{\tilde{y}\eta}{R+\tilde{d}}D_{11} + \frac{\tilde{y}\sin\delta}{\left(R+\tilde{d}\right)^2} + \frac{q\left(1-2\tilde{y}^2 D_{11}\right)}{\left(R+\tilde{d}\right)^2} - \frac{\cos\delta}{R} - qY_{11}\sin\delta\right\} = \frac{1}{2}\left\{-\frac{\tilde{d}q}{R\left(R+\tilde{d}\right)^2} + \frac{2q}{\left(R+\tilde{d}\right)^2} - \frac{2q^3}{R\left(R+\tilde{d}\right)^3} - \frac{q}{R\left(R+\tilde{d}\right)}\right\}$$

$$= \frac{q\left[-\tilde{d}\left(R+\tilde{d}\right) + 2R\left(R+\tilde{d}\right) - 2q^2 - \left(R+\tilde{d}\right)^2\right]}{2R\left(R+\tilde{d}\right)^3} = \frac{q\left[-R\left(R+\tilde{d}\right) + 2\left(R^2-\tilde{d}^2-q^2\right)\right]}{2R\left(R+\tilde{d}\right)^3} = -\frac{q}{(R+\tilde{d})^2}\left(\frac{1}{2} - \xi^2 D_{11}\right) = \boldsymbol{J_1}$$

$$J_6^y = \frac{\partial I_4}{\partial y} = \frac{1}{2}\frac{\partial}{\partial y}\frac{\xi\tilde{y}}{\left(R+\tilde{d}\right)^2} = \frac{\xi}{\left(R+\tilde{d}\right)^2}\left(\frac{1}{2} - q^2 D_{11}\right) = \boldsymbol{J_3}$$

$$J_3^y = \frac{\partial I_2}{\partial y} = \frac{\partial}{\partial y}\ln\left(R+\tilde{d}\right) + \frac{\partial I_3}{\partial y}\sin\delta = \tilde{y}D_{11} - \frac{q}{(R+\tilde{d})^2}\left(\frac{1}{2} - \xi^2 D_{11}\right)\sin\delta = \tilde{y}D_{11} - \frac{\tilde{y}}{(R+\tilde{d})^2}\left(\frac{1}{2} - \xi^2 D_{11}\right) = qY_{11} - \frac{2q}{R^2-\eta^2} - \boldsymbol{J_6}$$

$$J_1^y = -\frac{\partial I_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\xi}{R+\tilde{d}}\right)\cos\delta + \frac{\partial I_4}{\partial y}\sin\delta = \frac{\xi}{\left(R+\tilde{d}\right)^2}\left(\frac{1}{2} - q^2 D_{11}\right)\sin\delta = \xi Y_{11} + \boldsymbol{J_4}$$

For z-derivative

$$J_2^z = \frac{\partial}{\partial z}\left(\frac{\tilde{y}}{R+\tilde{d}}\right) = \tilde{y}D_{11}$$

$$J_5^z = \frac{\partial}{\partial z}\left(\frac{\xi}{R+\tilde{d}}\right) = \xi D_{11}$$

$$J_4^z = -\frac{\partial I_3}{\partial z} = -\frac{1}{2}\left\{-\frac{\sin\delta}{R+\tilde{d}} + \eta D_{11} + \frac{\tilde{y}}{\left(R+\tilde{d}\right)^2}\left(\cos\delta + \frac{2q}{R}\right) + \frac{\sin\delta}{R} - qY_{11}\cos\delta\right\} = -\frac{1}{2}\left\{\frac{\sin\delta}{R} - \frac{\sin\delta}{R+\tilde{d}} + \frac{\eta}{R\left(R+\tilde{d}\right)} + \frac{2\tilde{y}q}{R\left(R+\tilde{d}\right)^2}\right\}$$

$$= -\frac{1}{2}\left\{\frac{2\tilde{d}\sin\delta}{R\left(R+\tilde{d}\right)} + \frac{2q^2\sin\delta}{R\left(R+\tilde{d}\right)^2}\right\} = -\frac{\left(R+\tilde{d}\right)\tilde{d} + q^2}{R\left(R+\tilde{d}\right)^2}\sin\delta = -\frac{\sin\delta}{R+\tilde{d}}\left(1 - \xi^2 D_{11}\right) \equiv \boldsymbol{K_3}$$

$$J_6^z = -\frac{\partial I_4}{\partial z} = -\frac{1}{2}\frac{\partial}{\partial z}\frac{\xi\tilde{y}}{\left(R+\tilde{d}\right)^2} = -\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} \equiv \boldsymbol{K_4}$$

$$J_3^z = -\frac{\partial I_2}{\partial z} = -\frac{\partial}{\partial z}\ln\left(R+\tilde{d}\right) - \frac{\partial I_3}{\partial z}\sin\delta = \frac{1}{R} - \frac{1}{R+\tilde{d}}\left(1 - \xi^2 D_{11}\right) = \left(\tilde{d} + \frac{\xi^2}{R+\tilde{d}}\right)D_{11} \equiv \boldsymbol{K_2}$$

$$J_1^z = -\frac{\partial I_1}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\xi}{R+\tilde{d}}\right)\cos\delta + \frac{\partial I_4}{\partial z}\sin\delta = \frac{\xi\tilde{y}}{R+\tilde{d}}D_{11}\sin\delta = \frac{\xi q}{R+\tilde{d}}D_{11} \equiv \boldsymbol{K_1}$$

Namely

$$J_1^x = \boldsymbol{J_1} = -\frac{q}{(R+\tilde{d})^2}\left(\frac{1}{2} - \xi^2 D_{11}\right) \qquad J_1^y = \xi Y_{11} + \boldsymbol{J_4} \qquad J_1^z = \boldsymbol{K_1} = \frac{\xi q}{R+\tilde{d}}D_{11}$$

$$J_2^x = \boldsymbol{J_2} = \frac{\xi\tilde{y}}{R+\tilde{d}}D_{11} \qquad J_2^y = \frac{1}{R} + \boldsymbol{J_5} \qquad J_2^z = \tilde{y}D_{11}$$

$$J_3^x = \boldsymbol{J_3} = \frac{\xi}{\left(R+\tilde{d}\right)^2}\left(\frac{1}{2} - q^2 D_{11}\right) \qquad J_3^y = qY_{11} - \boldsymbol{J_6} \qquad J_3^z = \boldsymbol{K_2} = \left(\tilde{d} + \frac{\xi^2}{R+\tilde{d}}\right)D_{11}$$

$$J_4^x = \boldsymbol{J_4} = -\xi Y_{11} + \frac{\xi\sin\delta}{(R+\tilde{d})^2}\left(\frac{1}{2} - q^2 D_{11}\right) \qquad J_4^y = \boldsymbol{J_1} \qquad J_4^z = \boldsymbol{K_3} = -\frac{\sin\delta}{R+\tilde{d}}\left(1 - \xi^2 D_{11}\right)$$

$$J_5^x = \boldsymbol{J_5} = -\frac{1}{R+\tilde{d}}\left(1 - \xi^2 D_{11}\right) \qquad J_5^y = \boldsymbol{J_2} \qquad J_5^z = \xi D_{11}$$

$$J_6^x = \boldsymbol{J_6} = \frac{\tilde{y}}{(R+\tilde{d})^2}\left(\frac{1}{2} - \xi^2 D_{11}\right) \qquad J_6^y = \boldsymbol{J_3} \qquad J_6^z = \boldsymbol{K_4} = -\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11}$$

And there are following inter-relations

$$\boldsymbol{J_1} = J_5\cos\delta - \boldsymbol{J_6}\sin\delta \qquad \boldsymbol{K_1} = \boldsymbol{J_2}\sin\delta \qquad \boldsymbol{K_2} = \frac{1}{R} + \boldsymbol{K_3}\sin\delta$$

$$\boldsymbol{J_4} = -\xi Y_{11} - J_2\cos\delta + \boldsymbol{J_3}\sin\delta \qquad \boldsymbol{K_3} = \boldsymbol{J_5}\sin\delta \qquad \boldsymbol{K_4} = \xi Y_{11}\cos\delta - \boldsymbol{K_1}\sin\delta$$

Appendix : Table of Differentiation of Integrals

f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$
$\xi \quad (\leftrightarrow x)$	1	0	0
$\eta \quad (\leftrightarrow p = y\cos\delta + (c-z)\sin\delta)$	0	$\cos\delta$	$-\sin\delta$
$q \quad (\leftrightarrow q = y\sin\delta - (c-z)\cos\delta)$	0	$\sin\delta$	$\cos\delta$
$\tilde{y} \quad (= \eta\cos\delta + q\sin\delta)$	0	1	0
$\tilde{d} \quad (= \eta\sin\delta - q\cos\delta)$	0	0	-1
$h \quad (= q\cos\delta - z)$	0	$\sin\delta\cos\delta$	$-\sin^2\delta$
$\tilde{c} \quad (= \tilde{d} + z = \eta\sin\delta - h)$	0	0	0
$X \quad (= \sqrt{\xi^2+q^2})$	ξ/X	$q\sin\delta/X$	$q\cos\delta/X$
$R \quad (= \sqrt{\xi^2+\eta^2+q^2} = \sqrt{\xi^2+\tilde{y}^2+\tilde{d}^2})$	ξ/R	$\tilde{y}/R$	$-\tilde{d}/R$

f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$
$1/R$	$-\xi/R^3$	$-\tilde{y}/R^3$	$\tilde{d}/R^3$
$1/R^3$	$-3\xi/R^5$	$-3\tilde{y}/R^5$	$3\tilde{d}/R^5$
q/R	$-\xi q/R^3$	$\frac{\sin\delta}{R} - \frac{\tilde{y}q}{R^3}$	$\frac{\cos\delta}{R} + \frac{\tilde{d}q}{R^3}$
η/R^3	$-3\xi\eta/R^5$	$\frac{\cos\delta}{R^3} - \frac{3\tilde{y}\eta}{R^5}$	$-\frac{\sin\delta}{R^3} + \frac{3\tilde{d}\eta}{R^5}$
q/R^3	$-3\xi q/R^5$	$\frac{\sin\delta}{R^3} - \frac{3\tilde{y}q}{R^5}$	$\frac{\cos\delta}{R^3} + \frac{3\tilde{d}q}{R^5}$
$\ln(R+\xi)$	$1/R$	$\tilde{y}X_{11}$	$-\tilde{d}X_{11}$
$\ln(R+\eta)$	ξY_{11}	$\frac{\cos\delta}{R} + qY_{11}\sin\delta$	$-\frac{\sin\delta}{R} + qY_{11}\cos\delta$
$\ln(R+\tilde{d})$	ξD_{11}	$\tilde{y}D_{11}$	$-1/R$
X_{11}	$-1/R^3$	$-\tilde{y}X_{32}$	$\tilde{d}X_{32}$
X_{32}	$-3/R^5$	$-\tilde{y}X_{53}$	$\tilde{d}X_{53}$
ηX_{11}	$-\eta/R^3$	$X_{11}\cos\delta - \tilde{y}\eta X_{32}$	$-X_{11}\sin\delta + \tilde{d}\eta X_{32}$
$\tilde{y}X_{11}$	$-\tilde{y}/R^3$	$X_{11} - \tilde{y}^2X_{32}$	$\tilde{y}\tilde{d}X_{32}$
$\tilde{d}X_{11}$	$-\tilde{d}/R^3$	$-\tilde{y}\tilde{d}X_{32}$	$-X_{11} + \tilde{d}^2X_{32}$
ηqX_{11}	$-\eta q/R^3$	$(2\eta\sin\delta - \tilde{d})X_{11} - \tilde{y}\eta qX_{32}$	$(2\eta\cos\delta - \tilde{y})X_{11} + \tilde{d}\eta qX_{32}$
q^2X_{11}	$-q^2/R^3$	$q(2X_{11}\sin\delta - \tilde{y}qX_{32})$	$q(2X_{11}\cos\delta + \tilde{d}qX_{32})$
ηqX_{32}	$-3\eta q/R^5$	$(2\eta\sin\delta - \tilde{d})X_{32} - \tilde{y}\eta qX_{53}$	$(2\eta\cos\delta - \tilde{y})X_{32} + \tilde{d}\eta qX_{53}$
q^2X_{32}	$-3q^2/R^5$	$q(2X_{32}\sin\delta - \tilde{y}qX_{53})$	$q(2X_{32}\cos\delta + \tilde{d}qX_{53})$
ηq^2X_{32}	$-3\eta q^2/R^5$	$q[(3\eta\sin\delta - \tilde{d})X_{32} - \tilde{y}\eta qX_{53}]$	$-q[(3\eta\cos\delta - \tilde{y})X_{32} - \tilde{d}\eta qX_{53}]$
Y_{11}	$-\xi Y_{32}$	$-\frac{\cos\delta}{R^3} - qY_{32}\sin\delta \quad (*1)$	$\frac{\sin\delta}{R^3} - qY_{32}\cos\delta$
Y_{32}	$-\xi Y_{53}$	$-\frac{3\cos\delta}{R^5} - qY_{53}\sin\delta \quad (*2)$	$\frac{3\sin\delta}{R^5} - qY_{53}\cos\delta$
ξY_{11}	Y_0	$-\xi\left(\frac{\cos\delta}{R^3} + qY_{32}\sin\delta\right)$	$\xi\left(\frac{\sin\delta}{R^3} - qY_{32}\cos\delta\right)$
qY_{11}	$-\xi qY_{32}$	$\frac{\tilde{d}}{R^3} - Y_0\sin\delta \quad (*3)$	$\frac{\tilde{y}}{R^3} - Y_0\cos\delta$
ξqY_{11}	qY_0	$\xi\left(\frac{\tilde{d}}{R^3} - Y_0\sin\delta\right)$	$\xi\left(\frac{\tilde{y}}{R^3} - Y_0\cos\delta\right)$
q^2Y_{11}	$-\xi q^2Y_{32}$	$q\left(\frac{\tilde{d}}{R^3} + \xi^2Y_{32}\sin\delta\right) \quad (*4)$	$q\left(\frac{\tilde{y}}{R^3} + \xi^2Y_{32}\cos\delta\right)$
	As an alternate,	$(2\eta\sin\delta - \tilde{d}) = (\tilde{d} + 2q\cos\delta)$ and	$(2\eta\cos\delta - \tilde{y}) = (\tilde{y} - 2q\sin\delta)$

f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$
Z_{32}	$-\xi Z_{53}$	$-\frac{3\tilde{c}}{R^5}\cos\delta-(Y_{32}\cos\delta+qZ_{53})\sin\delta$ (∗5)	$\frac{3\tilde{c}}{R^5}\sin\delta+Y_{32}\sin^2\delta-qZ_{53}\cos\delta$ (∗5)
$\xi^2 Z_{32}$	$\xi(Z_{32}+Z_0)$	$-\xi^2\left\{\frac{3\tilde{c}}{R^5}\cos\delta+(Y_{32}\cos\delta+qZ_{53})\sin\delta\right\}$	$\xi^2\left\{\frac{3\tilde{c}}{R^5}\sin\delta+Y_{32}\sin^2\delta-qZ_{53}\cos\delta\right\}$
$q^2 Z_{32}$	$-\xi q^2 Z_{53}$	$q\left\{\frac{3\tilde{c}\tilde{d}}{R^5}-(zY_{32}+Z_0)\sin\delta\right\}$ (∗6)	$q\left\{\frac{3\tilde{c}\tilde{y}}{R^5}+qY_{32}-(zY_{32}+Z_0)\cos\delta\right\}$
$\xi q Z_{32}$	qZ_0	$\xi\left\{\frac{3\tilde{c}\tilde{d}}{R^5}-(zY_{32}+Z_{32}+Z_0)\sin\delta\right\}$ (∗7)	$\xi\left\{\frac{3\tilde{c}\tilde{y}}{R^5}+qY_{32}-(zY_{32}+Z_{32}+Z_0)\cos\delta\right\}$
$\frac{1}{R+\tilde{d}}$	$-\frac{\xi}{R+\tilde{d}}D_{11}$	$-\frac{\tilde{y}}{R+\tilde{d}}D_{11}$	D_{11}
$\frac{\xi}{R+\tilde{d}}$	$\left(\tilde{d}+\frac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11}$ (∗8)	$-\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11}$	ξD_{11}
$\frac{\eta}{R+\tilde{d}}$	$-\frac{\xi\eta}{R+\tilde{d}}D_{11}$	$\frac{\cos\delta}{R+\tilde{d}}-\frac{\tilde{y}\eta}{R+\tilde{d}}D_{11}$	$-\frac{\sin\delta}{R+\tilde{d}}+\eta D_{11}$
$\frac{\tilde{y}}{R+\tilde{d}}$	$-\frac{\xi\tilde{y}}{R+\tilde{d}}D_{11}$	$\frac{1}{R+\tilde{d}}-\frac{\tilde{y}^2}{R+\tilde{d}}D_{11}$	$\tilde{y}D_{11}$
$\frac{\xi\tilde{y}}{(R+\tilde{d})^2}$	$\frac{\tilde{y}(1-2\xi^2D_{11})}{(R+\tilde{d})^2}$	$\frac{\xi(1-2\tilde{y}^2D_{11})}{(R+\tilde{d})^2}$	$\frac{2\xi\tilde{y}}{R+\tilde{d}}D_{11}$
$\frac{\tilde{y}q}{(R+\tilde{d})^2}$	$-\frac{2\xi\tilde{y}q}{(R+\tilde{d})^2}D_{11}$	$\frac{\tilde{y}\sin\delta+q(1-2\tilde{y}^2D_{11})}{(R+\tilde{d})^2}$	$\frac{\tilde{y}}{(R+\tilde{d})^2}\left(\cos\delta+\frac{2q}{R}\right)$
$\Theta=\tan^{-1}\frac{\xi\eta}{qR}$	$-qY_{11}$ (∗9)	$\tilde{d}X_{11}+\xi Y_{11}\sin\delta$ (∗9)	$\tilde{y}X_{11}+\xi Y_{11}\cos\delta$
	$\frac{1}{2}(qY_{11}-\tilde{y}D_{11})$ (∗10)	$\frac{\xi}{2}(D_{11}-Y_{11}\sin\delta)$ (∗10)	$-\frac{1}{2}\xi Y_{11}\cos\delta$ (∗10)

$\tan^{-1}\frac{\eta(X+q\cos\delta)+X(R+X)\sin\delta}{\xi(R+X)\cos\delta}$ $h=q\cos\delta-z$ $\tilde{c}=\eta\sin\delta-h$

$$X_{11}=\frac{1}{R(R+\xi)}\quad X_{32}=\frac{2R+\xi}{R^3(R+\xi)^2}\quad X_{53}=\frac{8R^2+9R\xi+3\xi^2}{R^5(R+\xi)^3}\quad Z_{32}=\frac{\sin\delta}{R^3}-hY_{32}\quad Y_0=Y_{11}-\xi^2Y_{32}$$

$$D_{11}=\frac{1}{R(R+\tilde{d})}\quad Y_{11}=\frac{1}{R(R+\eta)}\quad Y_{32}=\frac{2R+\eta}{R^3(R+\eta)^2}\quad Y_{53}=\frac{8R^2+9R\eta+3\eta^2}{R^5(R+\eta)^3}\quad Z_{53}=\frac{3\sin\delta}{R^5}-hY_{53}\quad Z_0=Z_{32}-\xi^2Y_{53}$$

(∗1) $$\frac{\partial}{\partial y}Y_{11}=\frac{-1}{R^2(R+\eta)^2}\left[\frac{\tilde{y}}{R}(R+\eta)+R\left(\frac{\tilde{y}}{R}+\cos\delta\right)\right]=-\frac{\tilde{y}(2R+\eta)+R^2\cos\delta}{R^3(R+\eta)^2}=-\tilde{y}Y_{32}-\frac{\cos\delta}{R(R+\eta)^2}$$
$$=-(\eta\cos\delta+q\sin\delta)Y_{32}-\frac{\cos\delta}{R(R+\eta)^2}=-qY_{32}\sin\delta-\left(\eta Y_{32}+\frac{1}{R(R+\eta)^2}\right)\cos\delta=-qY_{32}\sin\delta-\frac{1}{R^3}\cos\delta$$

(∗2) $$\frac{\partial}{\partial y}Y_{32}=\frac{\left(2\frac{\tilde{y}}{R}+\cos\delta\right)R^3(R+\eta)^2-(2R+\eta)\left[3R^2\frac{\tilde{y}}{R}(R+\eta)^2+2R^3(R+\eta)\left(\frac{\tilde{y}}{R}+\cos\delta\right)\right]}{R^6(R+\eta)^4}=\frac{-\tilde{y}(8R^2+9R\eta+3\eta^2)-R^2(3R+\eta)\cos\delta}{R^5(R+\eta)^3}$$
$$=-(\eta\cos\delta+q\sin\delta)Y_{53}-\frac{3R+\eta}{R^3(R+\eta)^3}\cos\delta=-qY_{53}\sin\delta-\left(\eta Y_{53}+\frac{3R+\eta}{R^3(R+\eta)^3}\right)\cos\delta=-qY_{53}\sin\delta-\frac{3}{R^5}\cos\delta$$

(∗3) Since $(R^2-\eta^2)\frac{2R+\eta}{R^3(R+\eta)^2}=-\frac{\eta}{R^3}+\frac{2}{R(R+\eta)}$ → $(\xi^2+q^2)Y_{32}=-\frac{\eta}{R^3}+2Y_{11}$ → $(Y_{11}-\xi^2Y_{32})+(Y_{11}-q^2Y_{32})=\frac{\eta}{R^3}$
$$\frac{\partial}{\partial y}qY_{11}=Y_{11}\sin\delta-q\left(\frac{\cos\delta}{R^3}+qY_{32}\sin\delta\right)=(Y_{11}-q^2Y_{32})\sin\delta-\frac{q}{R^3}\cos\delta=\left[\frac{\eta}{R^3}-(Y_{11}-\xi^2Y_{32})\right]\sin\delta-\frac{q}{R^3}\cos\delta$$

(∗4) $$\frac{\partial}{\partial y}q^2Y_{11}=2qY_{11}\sin\delta-q^2\left(\frac{\cos\delta}{R^3}+qY_{32}\sin\delta\right)=q(2Y_{11}-q^2Y_{32})\sin\delta-\frac{q^2}{R^3}\cos\delta=q\left(\frac{\eta}{R^3}+\xi^2Y_{32}\right)\sin\delta-\frac{q^2}{R^3}\cos\delta$$

(∗5)
$$\frac{\partial}{\partial y}Z_{32}=-\left(\frac{3\tilde{y}}{R^5}+Y_{32}\cos\delta\right)\sin\delta+h\left(\frac{3\cos\delta}{R^5}+qY_{53}\sin\delta\right)=-\frac{3(\tilde{y}\sin\delta-h\cos\delta)}{R^5}-Y_{32}\sin\delta\cos\delta+qhY_{53}\sin\delta$$
$$=-\frac{3\tilde{c}}{R^5}\cos\delta-\frac{3q}{R^5}\sin^2\delta-Y_{32}\sin\delta\cos\delta+qhY_{53}\sin\delta=-\frac{3\tilde{c}}{R^5}\cos\delta-Y_{32}\sin\delta\cos\delta-q\sin\delta\left(\frac{3\sin\delta}{R^5}-hY_{53}\right)$$
$$\frac{\partial}{\partial z}Z_{32}=\left(\frac{3\tilde{d}}{R^5}+Y_{32}\sin\delta\right)\sin\delta-h\left(\frac{3\sin\delta}{R^5}-qY_{53}\cos\delta\right)=\frac{3(\tilde{d}-h)}{R^5}\sin\delta+Y_{32}\sin^2\delta+qhY_{53}\cos\delta$$
$$=\frac{3\tilde{c}}{R^5}\sin\delta-\frac{3q}{R^5}\sin\delta\cos\delta+Y_{32}\sin^2\delta+qhY_{53}\cos\delta=\frac{3\tilde{c}}{R^5}\sin\delta+Y_{32}\sin^2\delta-q\cos\delta\left(\frac{3\sin\delta}{R^5}-hY_{53}\right)$$

(∗6) Since $(R^2-\eta^2)\dfrac{8R^2+9R\eta+3\eta^2}{R^5(R+\eta)^3} = -\dfrac{3\eta}{R^5}+\dfrac{4(2R+\eta)}{R^3(R+\eta)^2} \rightarrow (\xi^2+q^2)Y_{53} = -\dfrac{3\eta}{R^5}+4Y_{32} \rightarrow (3Y_{32}-\xi^2Y_{53})+(Y_{32}-q^2Y_{53}) = \dfrac{3\eta}{R^5}$

and $Z_{32}-q^2Z_{53} = \left(\dfrac{1}{R^3}-\dfrac{3q^2}{R^5}\right)\sin\delta - h\left[\dfrac{3\eta}{R^5}-(3Y_{32}-\xi^2Y_{53})\right] = \left(\dfrac{1}{R^3}-\dfrac{3q^2}{R^5}\right)\sin\delta - \dfrac{3\eta(\eta\sin\delta-\tilde{c})}{R^5}+h(3Y_{32}-\xi^2Y_{53})$

$$= \frac{3\tilde{c}\eta}{R^5}-\frac{2\sin\delta}{R^3}+\frac{3\xi^2\sin\delta}{R^5}+h(3Y_{32}-\xi^2Y_{53}) = \frac{3\tilde{c}\eta}{R^5}+hY_{32}-2\left(\frac{\sin\delta}{R^3}-hY_{32}\right)+\xi^2\left(\frac{3\sin\delta}{R^5}-Y_{53}\right) = \frac{3\tilde{c}\eta}{R^5}+hY_{32}-2Z_{32}+\xi^2Z_{53}$$

$$\frac{\partial}{\partial y}q^2Z_{32} = 2qZ_{32}\sin\delta - q^2\left(\frac{3\tilde{c}}{R^5}\cos\delta+Y_{32}\sin\delta\cos\delta+qZ_{53}\sin\delta\right) = q\left\{-\frac{3\tilde{c}q}{R^5}\cos\delta - qY_{32}\sin\delta\cos\delta+(2Z_{32}-q^2Z_{53})\sin\delta\right\}$$

$$= q\left\{-\frac{3\tilde{c}q}{R^5}\cos\delta+\frac{3\tilde{c}\eta}{R^5}\sin\delta+[(h-q\cos\delta)Y_{32}-Z_{32}+\xi^2Z_{53}]\sin\delta\right\} = q\left\{\frac{3\tilde{c}\tilde{d}}{R^5}-[zY_{32}+Z_0]\sin\delta\right\}$$

(∗7) $\dfrac{\partial}{\partial y}\xi qZ_{32} = \xi\left\{Z_{32}\sin\delta - q\left(\dfrac{3\tilde{c}}{R^5}\cos\delta+Y_{32}\sin\delta\cos\delta+qZ_{53}\sin\delta\right)\right\} = \xi\left\{-\dfrac{3\tilde{c}q}{R^5}\cos\delta - qY_{32}\sin\delta\cos\delta+(Z_{32}-q^2Z_{53})\sin\delta\right\}$

$$= \xi\left\{-\frac{3\tilde{c}q}{R^5}\cos\delta+\frac{3\tilde{c}\eta}{R^5}\sin\delta+[(h-q\cos\delta)Y_{32}-2Z_{32}+\xi^2Z_{53}]\sin\delta\right\} = \xi\left\{\frac{3\tilde{c}\tilde{d}}{R^5}-[zY_{32}+Z_{32}+Z_0]\sin\delta\right\}$$

(∗8) $\dfrac{\partial}{\partial\xi}\dfrac{\xi}{R+\tilde{d}} = \dfrac{1}{R+\tilde{d}}-\dfrac{\xi^2}{R(R+\tilde{d})^2} = \dfrac{R(R+\tilde{d})-\xi^2}{R(R+\tilde{d})^2} = \dfrac{R\tilde{d}+\tilde{y}^2+\tilde{d}^2}{R(R+\tilde{d})^2} = \dfrac{(R+\tilde{d})\tilde{d}+\tilde{y}^2}{R(R+\tilde{d})^2} = \left(\tilde{d}+\dfrac{\tilde{y}^2}{R+\tilde{d}}\right)D_{11}$

(∗9)

$$\frac{\partial}{\partial\xi}\tan^{-1}\frac{\xi\eta}{qR} = \frac{q^2R^2}{\xi^2\eta^2+q^2R^2}\frac{\eta qR-\xi^2\eta q/R}{q^2R^2} = \frac{\eta q(R^2-\xi^2)}{R(\xi^2+q^2)(\eta^2+q^2)} = \frac{\eta q}{R(R^2-\eta^2)} = \frac{q}{R^2-\eta^2}-\frac{q}{R(R+\eta)} = -qY_{11}$$

$$\frac{\partial}{\partial y}\tan^{-1}\frac{\xi\eta}{qR} = \frac{\xi qR\cos\delta-\xi\eta(R\sin\delta+q\tilde{y}/R)}{\xi^2\eta^2+q^2R^2} = \frac{\xi q(R^2-\eta^2)\cos\delta-\xi\eta(R^2+q^2)\sin\delta}{R(\xi^2+q^2)(\eta^2+q^2)} = \frac{\xi q\cos\delta}{R(R^2-\xi^2)}-\frac{\xi\eta\sin\delta}{R(R^2-\xi^2)}-\frac{\xi\eta\sin\delta}{R(R^2-\eta^2)}$$

$$= \frac{q\cos\delta}{R^2-\xi^2}-\frac{q\cos\delta}{R(R+\xi)}-\frac{\eta\sin\delta}{R^2-\xi^2}+\frac{\eta\sin\delta}{R(R+\xi)}-\frac{\xi\sin\delta}{R^2-\eta^2}+\frac{\xi\sin\delta}{R(R+\eta)} = \tilde{d}X_{11}+\xi Y_{11}\sin\delta$$

(∗10) It was shown in "Derivation of Table 6", $\displaystyle\iint I_4^0 d\eta d\xi = \int\frac{\tilde{y}}{R(R+\tilde{d})}d\eta = \frac{1}{\cos\delta}[\ln(R+\tilde{d})-\sin\delta\ln(R+\eta)]$

and $\displaystyle\iint I_5^0 d\xi d\eta = \int\frac{\xi}{R(R+\tilde{d})}d\eta = \frac{2}{\cos\delta}\tan^{-1}\frac{\eta(X+q\cos\delta)+X(R+X)\sin\delta}{\xi(R+X)\cos\delta}$

As an alternative $\displaystyle\iint I_5^0 d\eta d\xi = \int\frac{1}{\cos\delta}\left(\frac{q}{R(R+\eta)}-\frac{\tilde{y}}{R(R+\tilde{d})}\right)d\xi = \frac{1}{\cos\delta}\left(\tan^{-1}\frac{\xi\tilde{d}}{\tilde{y}R}-\tan^{-1}\frac{\xi}{\tilde{y}}-\tan^{-1}\frac{\xi\eta}{qR}\right)$

Because $\displaystyle\int I_5^0\,d\eta = \int\left(\frac{1}{R(R+\tilde{d})}-\xi^2\frac{2R+\tilde{d}}{R^3(R+\tilde{d})^2}\right)d\eta = \int\left[\frac{\tilde{d}}{R^3}-\left(\frac{1}{R(R+\tilde{d})}-\tilde{y}^2\frac{2R+\tilde{d}}{R^3(R+\tilde{d})^2}\right)\right]d\eta = \int\left[\frac{\tilde{d}}{R^3}-\frac{\partial}{\partial\tilde{y}}\frac{\tilde{y}}{R(R+\tilde{d})}\right]d\eta$

$$= \int\frac{\eta\sin\delta-q\cos\delta}{R^3}d\eta-\frac{\partial}{\partial\tilde{y}}\int\frac{\tilde{y}}{R(R+\tilde{d})}d\eta = -\frac{\sin\delta}{R}+\frac{q\cos\delta}{R(R+\eta)}-\frac{1}{\cos\delta}\frac{\partial}{\partial\tilde{y}}[\ln(R+\tilde{d})-\sin\delta\ln(R+\eta)]$$

$$= -\frac{\sin\delta}{R}+\frac{q\cos\delta}{R(R+\eta)}-\frac{1}{\cos\delta}\left[\frac{\tilde{y}}{R(R+\tilde{d})}-\sin\delta\left(\frac{\cos\delta}{R}+\frac{q\sin\delta}{R(R+\eta)}\right)\right] = \frac{1}{\cos\delta}\left(\frac{q}{R(R+\eta)}-\frac{\tilde{y}}{R(R+\tilde{d})}\right)$$

and

$$\int\frac{q}{R(R+\eta)}d\xi = q\int\left(\frac{1}{R^2-\eta^2}-\frac{\eta}{R(R^2-\eta^2)}\right)d\xi = \tan^{-1}\frac{\xi}{q}-\tan^{-1}\frac{\xi\eta}{qR}$$

$$\int\frac{\tilde{y}}{R(R+\tilde{d})}d\xi = \tilde{y}\int\left(\frac{1}{R^2-\tilde{d}^2}-\frac{\tilde{d}}{R(R^2-\tilde{d}^2)}\right)d\xi = \tan^{-1}\frac{\xi}{\tilde{y}}-\tan^{-1}\frac{\xi\tilde{d}}{\tilde{y}R}$$

Therefore

$$\frac{\partial}{\partial\xi}\tan^{-1}\frac{\eta(X+q\cos\delta)+X(R+X)\sin\delta}{\xi(R+X)\cos\delta} = \frac{1}{2}\frac{\partial}{\partial\xi}\left(\tan^{-1}\frac{\xi\tilde{d}}{\tilde{y}R}-\tan^{-1}\frac{\xi}{\tilde{y}}-\tan^{-1}\frac{\xi\eta}{qR}\right) = \frac{1}{2}\left[\frac{\tilde{y}\tilde{d}(R^2-\xi^2)}{R(\tilde{y}^2+\xi^2)(\tilde{y}^2+\tilde{d}^2)}-\frac{\tilde{y}}{R^2-\tilde{d}^2}+qY_{11}\right]$$

$$= \frac{1}{2}\left[\left(\frac{\tilde{y}}{R^2-\tilde{d}^2}-\frac{\tilde{y}}{R(R+\tilde{d})}\right)-\frac{\tilde{y}}{R^2-\tilde{d}^2}+qY_{11}\right] = \frac{1}{2}(qY_{11}-\tilde{y}D_{11})$$

$$\frac{\partial}{\partial y}\tan^{-1}\frac{\eta(X+q\cos\delta)+X(R+X)\sin\delta}{\xi(R+X)\cos\delta} = \frac{1}{2}\frac{\partial}{\partial y}\left(\tan^{-1}\frac{\xi\tilde{d}}{\tilde{y}R}-\tan^{-1}\frac{\xi}{\tilde{y}}-\tan^{-1}\frac{\xi\eta}{qR}\right) = \frac{1}{2}\left[\frac{-\xi\tilde{d}(R^2+\tilde{y}^2)}{R(\tilde{y}^2+\xi^2)(\tilde{y}^2+\tilde{d}^2)}+\frac{\xi}{\tilde{y}^2+\xi^2}-(\tilde{d}X_{11}+\xi Y_{11}\sin\delta)\right]$$

$$= \frac{1}{2}\left[\frac{-\xi\tilde{d}}{R(R^2-\tilde{d}^2)}+\frac{-\xi\tilde{d}}{R(R^2-\xi^2)}+\frac{\xi}{R^2-\tilde{d}^2}-(\tilde{d}X_{11}+\xi Y_{11}\sin\delta)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\xi}{R(R+\tilde{d})}-\frac{\xi}{R^2-\tilde{d}^2}\right)+\left(\frac{\tilde{d}}{R(R+\xi)}-\frac{\tilde{d}}{R^2-\xi^2}\right)+\frac{\xi}{R^2-\tilde{d}^2}-(\tilde{d}X_{11}+\xi Y_{11}\sin\delta)\right] = \frac{1}{2}(\xi D_{11}-\xi Y_{11}\sin\delta)$$

$$\frac{\partial}{\partial z}\tan^{-1}\frac{\eta(X+q\cos\delta)+X(R+X)\sin\delta}{\xi(R+X)\cos\delta} = \frac{1}{2}\frac{\partial}{\partial z}\left(\tan^{-1}\frac{\xi\tilde{d}}{\tilde{y}R}-\tan^{-1}\frac{\xi}{\tilde{y}}-\tan^{-1}\frac{\xi\eta}{qR}\right) = \frac{1}{2}\left[\frac{-\xi\tilde{y}(R^2-\tilde{d}^2)}{R(\tilde{y}^2+\xi^2)(\tilde{y}^2+\tilde{d}^2)}-(\tilde{y}X_{11}+\xi Y_{11}\cos\delta)\right]$$

$$= \frac{1}{2}\left[\frac{-\xi\tilde{y}}{R(R^2-\xi^2)}-(\tilde{y}X_{11}+\xi Y_{11}\cos\delta)\right] = \frac{1}{2}\left[\frac{\tilde{y}}{R(R+\xi)}-\frac{\tilde{y}}{R^2-\xi^2}-(\tilde{y}X_{11}+\xi Y_{11}\cos\delta)\right] = -\frac{1}{2}\xi Y_{11}\cos\delta$$